Last Name: ______________________________

First Name: ______________________________

MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Final Exam (in-class portion)
January 22, 2003

Directions: You have three hours for this exam, though I have designed it to take less than the full amount of time. No calculators, notes, books, etc. are allowed. Please answer on the pages provided–there are blank page included for scratch work. Note that not all questions (and not all parts) are equally weighted.

Problem	Points	Score
1	28	
2	9	
3	16	
4	9	
5	14	
6	12	
Total	88	

1. True or False (28 points, 2 each)

T or **F** If V is a vector space and $S \subset V$ is a set of vectors that spans V, then S contains a basis for V.

T or **F** If U and V are vector spaces over the same field F, then $(U \oplus V)/V \cong U$.

T or **F** If $A, B \in M_n(F)$, then $\det(AB) = \det(A) \cdot \det(B)$.

T or **F** $\operatorname{sgn}((134)(25)) = +1$

T or **F** The number of *odd* permutations of n elements is $n!/2$.

T or **F** Every alternating multilinear form $f : V^n \longrightarrow F$ is skew-symmetric.

T or **F** If $dim(V) = m$ and $f : V^n \longrightarrow F$ is an alternating form with $n > m$, then $f = 0$.

T or **F** If $A, B \in M_n(F)$ and $\operatorname{Spec}(A) = \operatorname{Spec}(B)$, then there is some invertible $S \in M_n(F)$ such that $A = SBS^{-1}$.

T or **F** If V is a normed vector space with norm $|| \cdot ||$, then the function $d(x, y) = ||x - y||$ defines a metric on V.

T or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous and $S \subset \mathbb{R}^n$ is open, then $f(S)$ is open in $\mathbb{R}^m$.

T or **F** If $A \subset \mathbb{R}^n$ is open, then $A^\circ = A$, where A° is the interior of A.

T or **F** If $\{A_n\}_{n=1}^\infty$ is any collection of open sets in $\mathbb{R}^n$, then $\cap_{n=1}^\infty A_n$ is open.

T or **F** $\mathbb{Z}$ is closed as a subset of $\mathbb{R}$.

T or **F** Any three non-zero vectors in $\mathbb{R}^3$ may be turned into an orthonormal basis via the Gram-Schmidt Orthogonalization Process.

2. The Shift Operator (9 points, 2/3/4)

Let $\ell = \{(a_0, a_1, a_2, \ldots) | a_i \in \mathbb{R}, \forall i\}$ be the vector space of all infinite sequences of real numbers. Consider the shift operator $S : \ell \longrightarrow \ell$ that acts as follows: $S(a_0, a_1, a_2, a_3, \ldots) = (a_1, a_2, a_3, \ldots)$.

(a) Define the new operator $T : \ell \longrightarrow \ell$ by the rule $T = S^2 - S$. Write down the transformation T explicitly in terms of a vector $\mathbf{v} = (a_0, a_1, a_2, \ldots)$.

(b) Show that every real number $\lambda \in \mathbb{R}$ is an eigenvalue for T by explicitly naming a non-zero eigenvector corresponding to it.

(c) Let $\ell_1$be the eigenspace for T corresponding to the eigenvalue 1. Find a basis for ℓ_1, and prove that it is a basis.

3. Cramer's Rule (16 points, 2/4/3/3/4)

Consider the vector space $V = F^n$ and the invertible linear transformation $A : V \to V$. If $\mathbf{b} \in V$ is some fixed vector, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$, given as follows:

$$\text{If } A = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix}, \text{ let } \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

$$\text{where } x_i = (\det A)^{-1} \cdot \det \begin{bmatrix} | & & | & | & | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_{i-1} & \mathbf{b} & \mathbf{v}_{i+1} & \cdots & \mathbf{v}_n \\ | & & | & | & | & & | \end{bmatrix}.$$

Prove and apply this result in the following steps:

(Hint: You might do part (e) first to get a feel for this problem.)

(a) Write $\mathbf{b}$ as a linear combination of the columns of A. (Why can this be done?)

(b) If $D : V^n \to F$ is the non-zero alternating form used to define the determinant, evaluate the expression $D(\mathbf{v}_1, \cdots, \mathbf{v}_{i-1}, \mathbf{b}, \mathbf{v}_{i+1}, \cdots, \mathbf{v}_n)$, in terms of your linear combination from part (a).

(c) Show that the vector $\mathbf{x}$ as defined above satisfies the equation $A\mathbf{x} = \mathbf{b}$.

(d) Show that this $\mathbf{x}$ is the *unique* solution to the equation $A\mathbf{x} = \mathbf{b}$.

(e) Use Cramer's Rule to solve the system of equations:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ & & y & + & 4z & = & 0 \\ x & & & - & 6z & = & -1 \end{array}$$

4. Open and Closed Sets in Euclidean Space (9 points, 3 each)

Let V be a Euclidean space.

(a) If $A \subset V$, define what it means for A to be an *open set*.

(b) If $B \subset V$ and $\mathbf{x} \in V$, define what it means for $\mathbf{x}$ to be a *limit point* of B.

(c) If $V = \mathbb{R}^2$ and $C = \{(\frac{1}{n}, \frac{1}{m}) | n, m \in \mathbb{N}\}$, then find ∂C, the *boundary* of C.

5. Continuity and the Topology Euclidean Space (14 points, 3/3/8)

Let $M_n(\mathbb{R})$ be the n^2-dimensional Euclidean space of $n \times n$ matrices with real entries, and take the inner product to be the usual one, where $M_n(\mathbb{R})$ is naturally isomorphic to $\mathbb{R}^{n^2}$.

(a) Define what it means for $f : \mathbb{R}^n \to \mathbb{R}^m$ to be continuous at $\mathbf{x} \in \mathbb{R}^n$.

(b) Let $SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) = 1\}$. Show that $SL_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$.

(c) In this part, for simplicity, we specialize to the case $n = 2$. Let X be the collection of diagonalizable matrices in $M_2(\mathbb{R})$. Is X open, closed, or neither? Explain.

6. Linear Transformations on Euclidean Space (12 points, 4 each)

Let $V = \mathbb{R}^3$, and let $W = \text{span}\{(1, 2, 2), (0, 3, 6)\}$.

(a) Find an orthonormal basis for W.

(b) In terms of the standard basis for V, write the matrix for the linear transformation $P : V \to V$ that is projection onto the subspace W.

(c) Describe the linear transformation $2P - I$ geometrically, and show that $(2P - I)^2 = I$.