Math 23a: Theoretical Linear Algebra
and Multivariable Calculus I

# FINAL EXAM

January 24, 2006

*Your name:* ____________________

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 15 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 15 | |
| 8 | 20 | |
| Total | 100 | |

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Please show all your work on this exam paper. Unless otherwise stated, you must show your work and clearly indicate your line of reasoning in order to get full credit. You can write on the back of the pages if you need extra paper.

**Problem 1**

Decide whether the following statements are True or False. (Note: There is no need to justify your answers, just circle T or F. You get +2 for every correct answer and -1 for every wrong answer.)

**T or F:** If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $n > m$, then Ker$L$ is not zero.

**T or F:** Every bounded set of real numbers has a least element.

**T or F:** If $V$ is a Euclidean space and $u, v \in V$, then $||u + v|| = ||u|| + ||v||$.

**T or F:** The columns of an $n \times n$ matrix are linearly independent if and only if $A$ is invertible.

**T or F:** The empty set is a vector space over any field.

**Problem 2**

Decide whether the following statements are True or False. For each statement provide a brief justification or give a counterexample.

(a) If $T : U \to V$ and $S : V \to W$ are both injective linear transformations, then $S \circ T$ is also injective.

**True or False:**

**Argument:**

(b) If $U$ is a subspace of a Euclidean space $V$, then $U \cap U^{\perp} = \{0\}$.

**True or False:**

**Argument:**

(c) If $V$ is a Euclidean space, then it is possible to find an orthonormal basis for $V$.

**True or False:**

**Argument:**

(d) Let $T : V \to W$ be a linear map and let $v_1, \ldots, v_n$ be linearly independent vectors in $V$. Then $T(v_1), \ldots, T(v_n)$ are linearly independent.

**True or False:**

**Argument:**

(e) If $U$ is a subspace of a finite dimensional vector space $V$ and $\dim U = \dim V$, then $U = V$.

**True or False:**

**Argument:**

**Problem 3**

Let $S \subset \mathrm{Mat}_{4\times 4}(\mathbb{R})$ be the set of all $4 \times 4$ hermitian (i.e. symmetric) real matrices $A$ such that $Av = 0$, where $v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Prove that $S$ is a subspace of $\mathrm{Mat}_{4\times 4}(\mathbb{R})$ and compute its dimension.

**Problem 4**

Let $A = \begin{bmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{bmatrix}$.

(a) For what (real) values of $x$ is $A$ invertible? Explain.
(b) For what (real) valies of $x$ is $A$ orthogonal? Explain.

**Problem 5**

Consider the space $\mathbb{R}^n$ with the standard dot product $\langle u|v\rangle = v^t u$, and let $A$ be an operator on $\mathbb{R}^n$ (i.e. an $n \times n$ matrix with real coefficients).

(a) Prove that any matrix $A$ can be written in a unique way as a sum of a symmetric matrix and a skew-symmetric matrix.

(b) Suppose $A$ is symmetric. Prove that $\langle Av|v\rangle = 0$ for every $v \in V$ if and only if $A = 0$.

(c) More in general, prove that $\langle Av|v\rangle = 0$ for every $v \in V$ if and only if $A + A^t = 0$ (i.e. $A$ is skew-symmetric).

**Problem 6**

In this problem you are asked to prove a special case of what we stated in class as Jordan's Theorem.

Let $V = \mathbb{R}^2$ and let $A : V \to V$. Suppose that the characteristic polynomial of $A$ is $p_A(\lambda) = (\alpha - \lambda)^2$.

(a) Show that exactly one of the following possibilities must hold:
- $A$ is diagonalizable. (what is the diagonalized form of $A$?)
- There is a basis of $V$ with respect to which

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$$

(b) Suppose $\alpha = 1$. Use part (a) and a change of basis to show that $(A - \mathbb{1})^2 = 0$.

(c) Again for $\alpha = 1$, use part $(b)$ to show that $Av - v \in V_1$ for every $v \in V$, where $V_1$ is the eigenspace corresponding to the eigenvalue 1.

**Problem 7**

(a) Let $V$ be a vector space. Give the general definition of an inner product on $V$.

(b) For an $n \times n$ matrix $A$, let $tr(A) = \sum_{i=1}^{n} a_{ii}$, the sum of diagonal entries. Show that $\langle A|B\rangle = tr(A^t B)$ defines an inner product on the space of $n \times n$ matrices with real entries.

**Problem 8**

Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$ with dimension $n$. Recall the definition of the dual space of $V$:

$$V^* \{f : V \to \mathbb{F} \mid f \text{ is linear}\}.$$

Find a *natural* isomorphism $(V^*)^* \simeq V$ (i.e. which does not depend on any choice of bases).