Math 23aTheoretical Linear Algebra and Multivariable Calculus I

MIDTERM EXAM 1 - PRACTICE EXAM

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Problem 1: (a) State the Principle of Mathematical Induction.
(b) Prove, by induction, that the following identity is true for every $n \geq 1$:

$$-1^2 + 2^2 - 3^2 + 4^2 - \cdots + (-1)^n n^2 = \frac{1}{2}(-1)^n n(n+1) .$$

Solution 1: (a) Suppose we have a statement $A(n)$ depending on a positive integer n. Then $A(n)$ is true for every $n \geq 1$, provided that the following happens:
1) A_1 is true,
2) Assuming A_n is true for an arbitrary n, it follows that A_{n+1} is true.
(b) Our statement $A(n)$ is the identity

$$-1^2 + 2^2 - 3^2 + 4^2 - \cdots + (-1)^n n^2 = \frac{1}{2}(-1)^n n(n+1) .$$

We do the two induction steps:
1) We show that $A(1)$ is true. Indeed the following identity id true (by direct inspection)

$$-1 = \frac{1}{2}(-1)^1 1(1+1)$$

2) We now assume that $A(n)$ is true for a fixed (arbitrary) n. Namely we assume

$$-1^2 + 2^2 - 3^2 + 4^2 - \cdots + (-1)^n n^2 = \frac{1}{2}(-1)^n n(n+1) .$$

If we add $(-1)^{n+1}(n+1)^2$ to both sides we get

$$\begin{aligned} & -1^2 + 2^2 - 3^2 + 4^2 - \cdots + (-1)^n n^2 + (-1)^{n+1}(n+1)^2 \\ & = \tfrac{1}{2}(-1)^n n(n+1) + (-1)^{n+1}(n+1)^2 \\ & = (-1)^{n+1}(n+1)\Big(-\tfrac{1}{2}n + n + 1\Big) = \tfrac{1}{2}(-1)^{n+1}(n+1)(n+2) , \end{aligned}$$

namely we proved that statement $A(n+1)$ is true.
According to the Principle of Mathematical Induction, we thus conclude that $A(n)$ is true for every n.

Problem 2: (a) Define what is an *order* relation.
(b) Define what is an *ordered* field.
(c) Prove that $\mathbb{Z}/5\mathbb{Z}$ is not an ordered field.

Solution 2: (a) An order relation $>$ on a set S is a relation satisfying the following axioms:
1) tricotomy: given a and b in S, exactly one of the following things happen:
$$a > b\ ,\quad b > a\ ,\quad a = b\ .$$
2) transitivity: if $a > b$ and $b > c$, then $a > c$,

(b) An ordered field is a field $\mathbb{F}$ together with an ordered relation $>$ such that the following axioms hold:
1) if a, b, c are elements of $\mathbb{F}$ and $b > c$, then $a + b > a + c$, 2) if a, b are elements of $\mathbb{F}$ such that $a > 0$ and $b > 0$, then $ab > 0$.

(c) Suppose, by contradiction, that $\mathbb{Z}/5\mathbb{Z}$ is an ordered field. In class we proved that in any ordered field $1 > 0$. Then in our case we have $[1] > [0]$, and by axiom 1) and transitivity of the order relation we get $[2] = [1] + [1] > [1] + [0] = [1] > [0]$, and again by axiom 1) and transitivity of the order relation we get $[3] = [2] + [1] > [2] + [0] = [2] > [0]$, and again by axiom 1) and transitivity of the order relation we get $[4] = [3] + [1] > [3] + [0] = [3] > [0]$, and again by axiom 1) and transitivity of the order relation we get $[5] = [4] + [1] > [4] + [0] = [4] > [0]$. So we proved $[5] > [0]$. But in $\mathbb{Z}/5\mathbb{Z}$ we have $[5] = [0]$, so we have a contradiction.

Problem 3: Which of the following subsets $U \subset V$ is a subspace? (Justify your answer)

(a) $V = \mathbb{R}^3\ ,\quad U = \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \alpha_1 = 1 \right\} \subset V.$

(b) $V = \mathbb{R}^3\ ,\quad U = \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \alpha_1 + 2\alpha_2 = 0 \right\} \subset V.$

(c) $V = \mathbb{R}^3\ ,\quad U = \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \alpha_1^2 = 2\alpha_2 \right\} \subset V.$

(d) V is the set of all functions $f : \mathbb{R} \to \mathbb{R}$, considered as a vector space over $\mathbb{Q}$, and $U = \big\{ f \in \mathrm{Fun}(\mathbb{R}) \;\big|\; f(1/2) \in \mathbb{Q} \big\} \subset V$.

Solution 3: (a) U is not a subspace. Indeed $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in U$, but $2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \notin U$

(b) U is a subspace. Indeed, take arbitrary $\lambda \in \mathbb{R}$, $a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \in U$ and $b = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \in U$, namely $\alpha_1 + 2\alpha_2 = 0$ and $\beta_1 + 2\beta_2 = 0$. Then we have
$$\lambda\alpha_1 + 2\lambda\alpha_2 = \lambda(\alpha_1 + 2\alpha_2) = 0,$$

which means that $\lambda a = \begin{bmatrix} \lambda\alpha_1 \\ \lambda\alpha_2 \\ \lambda\alpha_3 \end{bmatrix} \in U$, and we also have

$$(\alpha_1 + \beta_1) + 2(\alpha_2 + \beta_2) = (\alpha_1 + 2\alpha_2) + (\beta_1 + 2\beta_2) = 0 + 0 = 0,$$

which means that $a + b = \begin{bmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \end{bmatrix} \in U$. We thus proved that U is a subspace of V, since it's closed under addition and scalar multiplication.

(c) U is not a subspace, indeed $a = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \in U$ (since $2^2 = 2 \cdot 2$), but $2a = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} \notin U$ (since $4^2 = 16 \neq 2 \cdot 4 = 8$).

(d) U is a subspace. Indeed if $\alpha \in \mathbb{Q}$ and $f \in U$, namely $f(1/2) \in \mathbb{Q}$, then $\alpha f(1/2) \in \mathbb{Q}$, which means that the function αf belongs to U. Moreover, if $f, g \in U$, namely $f(1/2) \in \mathbb{Q}$ and $g(1/2) \in \mathbb{Q}$, then $f(1/2) + g(1/2) \in \mathbb{Q}$, which means that the function $f + g$ belongs to U. We thus proved that U is a subspace of V, since it's closed under addition and scalar multiplication.

Problem 4: Let V be a vector space such that $\dim V = n$.

(a) Let $v_1, \ldots, v_n$ be vectors of V such that $\operatorname{span}(v_1, \ldots, v_n) = V$. Prove that $(v_1, \ldots, v_n)$ is a basis of V.

(b) Let $v_1, \ldots, v_n$ be linearly independent vectors of V. Prove that $(v_1, \ldots, v_n)$ is a basis of V.

Solution 4: (a) We proved in class that any list of vectors $(v_1, \ldots, v_n)$ which spans V can be reduced to a basis of V. Namely we get a basis of V by a sublist of $(v_1, \ldots, v_n)$. But any basis of V consists of exactly n elements (since $\dim V = n$), so $(v_1, \ldots, v_n)$ needed to be a basis to start with.

(b) We proved in class that any list of linearly independent vectors in a finite dimensional vector space can be extended to a basis. Therefore we get a basis of V of the following type $(v_1, \ldots, v_n, w_1, \ldots, w_m)$. But any basis of V consists of exactly n elements (since $\dim V = n$), so none of the w's actually can appear and $(v_1, \ldots, v_n)$ needed to be a basis to start with.

Problem 5: Which of the following lists of vectors of $\mathbb{R}^2$ is linearly independent and which is not? (Justify your answer)

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Solution 5: (a) We proved in class that two vectors are linearly independent if and only if one of them is proportional to the other. So these vectors are linearly independent, since none of the vector is proportional to the other.

(b) Linearly independent, since none of the vector is proportional to the other.

(c) Linearly dependent, since $\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(d) Linearly dependent, since $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. (By what we proved in class, one of the equivalent definitions of linearly dependence is that one of the vectors of the list can be expressed as linear combination of the previous ones).