Math 23aTheoretical Linear Algebra and Multivariable Calculus I

MIDTERM EXAM 2 - PRACTICE EXAM

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Problem 1: Let U and V be subspaces of a vector space W. In this problem we want to prove the "Second Isomorphism Theorem":

$$U/(U \cap V) \ \simeq (U+V)/V \ .$$

(The "first isomorphism theorem" is $V/\mathrm{Ker}T \simeq \mathrm{Im}T$, which you proved in a Problem Set)

(a) Consider the linear transformation $\pi : U+V \to (U+V)/V$ defined by $\pi(w) = [w]$ for all $w \in U+V$. Describe its kernel.
(b) Consider the linear transformation $T : U \to (U+V)/V$ obtained by composing the identity map $U \subset U+V$ (also called "inclusion map") and π. Find the kernel of T.
(c) Prove that T is surjective.
(d) Deduce that $U/(U \cap V)$ and $(U+V)/V$ are isomorphic vector spaces using the first isomorphism theorem.

Solution: (a) $\mathrm{Ker}\pi = V \subset U+V$.
(b) $\mathrm{Ker}T = \{u \in U \,|\, \pi(u) = 0\} = \{u \in U \,|\, u \in V\} = U \cap V$.
(c) Let $[w] \in (U+V)/V$, be an arbitrary element, with $w = u+v \in U+V$. Clearly $[w] = [u]$ (since $w - u = v \in V$, so that $w \sim u$). Hence $[w] = T(u)$, which means that T is surjective.
(d) Since $T : U \to (U+V)/V$ is a linear map, by the first isomorphism theorem: $U/\mathrm{Ker}T \simeq \mathrm{Im}T$, which is what we wanted, since $\mathrm{Ker}T = U \cap V$ and $\mathrm{Im}T = (U+V)/V$.

Problem 2: Consider the vector space $\mathcal{P}$ of all polynomials in x with real coefficients. Let R, S, T be the operators on $\mathcal{P}$ which map the polynomial $P(x) = \sum_{k=0}^{n} c_k x^k \ \in \mathcal{P}$ to the polynomials $r(x)$, $s(x)$, $t(x)$ respectively, where

$$r(x) = P(0) \ , \quad s(x) = \sum_{k=1}^{n} c_k x^{k-1} \ , \quad t(x) = \sum_{k=0}^{n} c_k x^{k+1} \ .$$

(a) Let $P(x) = 2 + 3x - x^2 + x^3$. Find the polynomials $TS(P(x))$ and $RST(P(x)$.
(b) Determine the Kernel and the Image of each of the linear transformations R, S, T.

Solution: (a) $T(S(P(x))) = T(S(2+3x-x^2+x^3)) = T(3-x+x^2) = 3x - x^2 + x^3$, and $R(S(T(P(x)))) = R(S(T(2+3x-x^2+x^3))) = R(S(2x+3x^2-x^3+x^4)) = R(2+3x-x^2+x^3) = 2$.

(b) $\text{Ker}R = \{P(x) = \sum_{k=1}^{n} c_k x^k\}$, $\text{Im}R = \{c \in \mathbb{R}\} = \mathbb{R}$, $\text{Ker}S = \{c \in \mathbb{R}\} = \mathbb{R} = \text{Im}R$, $\text{Im}S = \{\sum_{k=0}^{n} d_k x^k\} = \mathcal{P}$, $\text{Ker}T = 0$, $\text{Im}T = \{P(x) = \sum_{k=1}^{n} d_k x^k\} = \text{Ker}R$.

Problem 3: A 4×4 matrix is called **symplectic** if $AJA^t = J$ (A^t is the transpose of the matrix A), where J is the matrix

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

(a) Verify that J is a symplectic matrix.
(b) Prove that if A is symplectic, than A is invertible and A^{-1} is symplectic.
(c) Prove that if A and B are symplectic, then AB is symplectic.
(d) Prove that if A is symplectic, then $\det A$ is either 1 or -1.

Solution: (a) By definition of J,

$$\begin{aligned} J^2 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = -\mathbb{1}. \end{aligned}$$

and

$$J^t = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}^t = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = -J$$

Hence $JJJ^t = (-\mathbb{1})(-J) = J$, which means that J is symplectic.

(b) Notice that $\det J = 1$. If A i symplectic, then $AJA^t = J$, hence $1 = \det J = \det(AJA^t) = \det(A)\det(J)\det(A^t) = (\det(A))^2$. Hence $\det(A) = \pm 1$ and A is invertible. Notice that $(A^{-1})^t = (A^t)^{-1}$ (Indeed $(A^{-1})^t A^t = (AA^{-1})^t = \mathbb{1}^t = \mathbb{1}$, and similarly $A^t(A^{-1})^t = (A^{-1}A)^t = \mathbb{1}^t = \mathbb{1}$, which means that $(A^{-1})^t = (A^t)^{-1}$). Hence, if we multiply both sides of the equation $AJA^t = J$ on the left by A^{-1} and on the right by $(A^{-1})^t$, we get $J = A^{-1}AJA^t(A^t)^{-1} = A^{-1}J(A^{-1})^t$, which means that A^{-1} is symplectic.

(c) $ABJ(AB)^t = ABJB^tA^t = A(BJB^t)A^t = AJA^t = J$.

(d) It was already proved in (b).

Problem 4: In this problem we will prove Cramer's Rule for solving systems of linear equations. Consider an arbitrary linear system of n equations in n

unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Let A be the matrix of the coefficients, let B be the column vector with entries $b_1, b_2, \ldots, b_n$, and let X be the column vector with entries $x_1, x_2, \ldots, x_n$. Assume $\det A \neq 0$.

(a) Write the formula for the inverse of the matrix A.

(b) Prove that $(x_1, x_2, \ldots, x_n)$ is a solution of the system of equation if and only if

$$X = A^{-1}B .$$

(c) Use the formula for A^{-1} to prove that x_i is the determinant of the matrix obtained from A by replacing column i by the column B:

$$x_i = \frac{1}{\det A} \det \begin{bmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ & & \cdots & & \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix} .$$

Solution: (a) $A^{-1} = \frac{1}{\det A} B$ where B is the matrix which, in position (i, j), has entry

$$b_{ij} = (-1)^{i+j} \det \begin{bmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1,i+1} & \cdots & a_{1n} \\ & \cdots & & & \cdots & \\ a_{j-1,1} & \cdots & a_{j-1,i-1} & a_{j-1,i+1} & \cdots & a_{j-1,n} \\ a_{j+1,1} & \cdots & a_{j+1,i-1} & a_{j+1,i+1} & \cdots & a_{j+1,n} \\ & \cdots & & & \cdots & \\ a_{n1} & \cdots & a_{n,i-1} & a_{n,i+1} & \cdots & a_{nn} \end{bmatrix} .$$

(b) Suppose $(x_1, x_2, \ldots, x_n)$ is a solution, namely $AX = B$. If we multiply both sides of this equation by A^{-1} on the left, we get $X = A^{-1}AX = A^{-1}B$. conversely, suppose $X = A^{-1}B$. If we multiply both sides of this equation by A on the left, we get $AX = AA^{-1}B = B$, namely $(x_1, x_2, \ldots, x_n)$ is a solution.

(c) From (b), x_i is obtained by multiplying row i of A^{-1} by the column B. Namely

$$x_i = \frac{1}{\det A}[b_{i1}, b_{i2}, \ldots, b_{in}] \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \frac{1}{\det A} \sum_{j=1}^{n} b_{ij} b_j$$

$$= \sum_{j=1}^{n} \frac{(-1)^{i+j} b_j}{\det A} \begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1,i+1} & \cdots & a_{1n} \\ & \cdots & & & \cdots & \\ a_{j-1,1} & \cdots & a_{j-1,i-1} & a_{j-1,i+1} & \cdots & a_{j-1,n} \\ a_{j+1,1} & \cdots & a_{j+1,i-1} & a_{j+1,i+1} & \cdots & a_{j+1,n} \\ & \cdots & & & \cdots & \\ a_{n1} & \cdots & a_{n,i-1} & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix}$$

But now on the right hand side we have $\frac{1}{\det A}$ times the formula for the j-th column expansion of the following determinant

$$\det \begin{bmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ & & \cdots & & \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix},$$

which is what we wanted.

Problem 5: Find an example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ which has no eigenvalues.

Solution: T can the linear transformation given by the following matrix

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Indeed the characteristic polynomial is $P(\lambda) = \det(A - \lambda \mathbb{1}) = (1 + \lambda^2)^2$, which has no real roots.