Math 23aTheoretical Linear Algebra and Multivariable Calculus I

MIDTERM EXAM 2 - PRACTICE EXAM

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Problem 1: Let U and V be subspaces of a vector space W. In this problem we want to prove the "Second Isomorphism Theorem":

$$U/(U \cap V) \simeq (U+V)/V \ .$$

(The "first isomorphism theorem" is $V/\mathrm{Ker}T \simeq \mathrm{Im}T$, which you proved in a Problem Set)

(a) Consider the linear transformation $\pi : U+V \to (U+V)/V$ defined by $\pi(w) = [w]$ for all $w \in U+V$. Describe its kernel.
(b) Consider the linear transformation $T : U \to (U+V)/V$ obtained by composing the identity map $U \subset U+V$ (also called "inclusion map") and π. Find the kernel of T.
(c) Prove that T is surjective.
(d) Deduce that $U/(U \cap V)$ and $(U+V)/V$ are isomorphic vector spaces using the first isomorphism theorem.

Problem 2: Consider the vector space $\mathcal{P}$ of all polynomials in x with real coefficients. Let R, S, T be the operators on $\mathcal{P}$ which map the polynomial $P(x) = \sum_{k=0}^{n} c_k x^k \in \mathcal{P}$ to the polynomials $r(x)$, $s(x)$, $t(x)$ respectively, where

$$r(x) = P(0) \ , \quad s(x) = \sum_{k=1}^{n} c_k x^{k-1} \ , \quad t(x) = \sum_{k=0}^{n} c_k x^{k+1} \ .$$

(a) Let $P(x) = 2 + 3x - x^2 + x^3$. Find the polynomials $TS(P(x))$ and $RST(P(x)$.
(b) Determine the Kernel and the Image of each of the linear transformations R, S, T.

Problem 3: A 4×4 matrix is called **symplectic** if $AJA^t = J$ (A^t is the transpose of the matrix A), where J is the matrix

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} .$$

(a) Verify that J is a symplectic matrix.
(b) Prove that if A is symplectic, than A is invertible and A^{-1} is symplectic.
(c) Prove that if A and B are symplectic, then AB is symplectic.
(d) Prove that if A is symplectic, then $\det A$ is either 1 or -1.

Problem 4: In this problem we will prove Cramer's Rule for solving systems of linear equations. Consider an arbitrary linear system of n equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Let A be the matrix of the coefficients, let B be the column vector with entries $b_1, b_2, \ldots, b_n$, and let X be the column vector with entries $x_1, x_2, \ldots, x_n$. Assume $\det A \neq 0$.

(a) Write the formula for the inverse of the matrix A.

(b) Prove that $(x_1, x_2, \ldots, x_n)$ is a solution of the system of equation if and only if

$$X = A^{-1}B .$$

(c) Use the formula for A^{-1} to prove that x_i is the determinant of the matrix obtained from A by replacing column i by the column B:

$$x_i = \frac{1}{\det A} \det \begin{bmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ & & \cdots & & \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix} .$$

Problem 5: Find an example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ which has no eigenvalues.