Problem Set 5, Problem 4

a) Need to show: if $[v]=[w]$, then $\lambda[v]=\lambda[w]$.

$[v]=[w] \Leftrightarrow v-w=u,\ u\in U \Rightarrow \lambda v-\lambda w=\lambda u,\ \lambda u\in U \Leftrightarrow [\lambda v]=[\lambda w] \Leftrightarrow$

$\Leftrightarrow \lambda[v]=\lambda[w]$.

U is a subspace, closed under $\cdot\lambda$

Need to show: if $[v]=[w]$, $[v']=[w']$, then $[v+v']=[w+w']$.

$[v]=[w],\ [v']=[w'] \Leftrightarrow v-w=u,\ u\in U,\ v'-w'=u',\ u'\in U \Leftrightarrow$

$(v-w)+(v'-w')=u+u',\ (v+v')-(w+w')=u+u',\ u+u'\in U \Leftrightarrow$

$[v+v']=[w+w']$

U is a subspace.

b) Need to show: closure under addition and scalar multiplication; axioms

- if $[v]\in V/U \Rightarrow [v]=\{v+u \mid u\in U\} \Rightarrow \lambda[v]=[\lambda v]=\{\lambda v+\lambda u \mid u\in U\} \Rightarrow$
  $\lambda[v]=\{\lambda v+u' \mid u'\in U\} \Rightarrow \lambda[v]\in V/U$.
- closure under addition is analogous.
- commutative law: $[v]+[w]=[v+w]=[w+v]=[w]+[v]$
- other axioms are analogous (you have done this 1000 times already :))
- multiplicative identity: $1\in F$
- additive identity: $[0]$
- additive inverse of $[x]$ is $[-x]$.

c) (VERY COOL)

Define $T: V\to V/U$, $T(v)=[v]$

- T is linear
  $T(v+w)=[v+w]=[v]+[w]=T(v)+T(w)$
  $T(\lambda v)=[\lambda v]=\lambda[v]=\lambda T(v)$
- T is surjective
  For each $[v]\in V/U$, $[v]=T(v)$, so $[v]\in \operatorname{Im} T$. $\Rightarrow \operatorname{Im} T=V/U$
  In other words, $V/U \subset \operatorname{Im} T$, $\operatorname{Im} T\subset V/U$

So we can apply the rank-nullity theorem:

(1) $\dim V=\dim(\operatorname{Im} T)+\dim(\ker T)=\dim(V/U)+\dim(\ker$

Observe that $Tv=[0] \Leftrightarrow v=0+u,\ u\in U \Leftrightarrow v\in U$.

So $\ker T=U$. Substituting in (1), we get:

$\dim V=\dim(V/U)+\dim U$, which is what we wanted to prove.

d) We will show that $\{[x^3], [x^4], \ldots, [x^n]\}$ is a basis for P_n / P_2. From part c) we know that $\dim P_n / P_2 = \dim P_n - \dim P_2 = n+1-3 = n-2$. Since there are $n-2$ vectors in our set, it is enough to show that they are linearly independent in P_n / P_2 and it will follow that they form a basis.

Let $\lambda_3, \ldots, \lambda_n$ be such that $\lambda_3 [x^3] + \lambda_4 [x^4] + \ldots + \lambda_n [x^n] = [0]$.

We will show that $\lambda_3 = \lambda_4 = \ldots = \lambda_n = 0$.

By our definition of addition and scalar multiplication, we get

$$[\lambda_3 x^3 + \lambda_4 x^4 + \ldots + \lambda_n x^n] = [0]$$

$\Leftrightarrow \lambda_3 x^3 + \lambda_4 x^4 + \ldots + \lambda_n x^n \in P_2$, so the degree of $\lambda_3 x^3 + \ldots + \lambda_n x^n$ is less than or equal to 2, so $\lambda_3 = \lambda_4 = \ldots = \lambda_n = 0$. So what we have is indeed a basis.

still life in a square ;)