1. Solution to problem 4, with many thanks to William Phan.

4. (a) For any vectors $v, u \in V, \lambda \in R, v = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $u = \{\beta_1, \beta_2, \ldots, \beta_n\}$, define addition and multiplication as:

$$
\begin{aligned}
u + v &= \{\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n\} \\
\lambda \cdot v &= \{\lambda \cdot \alpha_1, \lambda \cdot \alpha_2, \ldots, \lambda \cdot \alpha_n\}
\end{aligned}
$$

To prove that V is a vector space over R, we need to show that:

i. *V is commutative*

$$
\begin{aligned}
u + v &= \{\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n\} \\
&= \{\beta_1 + \alpha_1, \beta_2 + \alpha_2, \ldots, \beta_n + \alpha_n\} \qquad \text{by commutative law on } R \\
&= v + u
\end{aligned}
$$

$$
\begin{aligned}
\lambda \cdot v &= \{\lambda\alpha_1, \ \lambda\alpha_2, \ \ldots, \ \lambda\alpha_n\} \\
&= \{\alpha_1\lambda, \ \alpha_2\lambda, \ \ldots, \ \alpha_n\lambda\} \\
&= v \cdot \lambda
\end{aligned}
$$

Therefore, V is commutative.

ii. *V is associative*

For $w = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$

$$
\begin{aligned}
u + (v + w) &= u + \{\beta_1 + \gamma_1, \beta_2 + \gamma_2, \ldots, \beta_n + \gamma_n\} \\
&= \{\alpha + (\beta_1 + \gamma_1), \alpha_2 + (\beta_2 + \gamma_2), \ldots, \alpha_n + (\beta_n + \gamma_n)\} \\
&= \{(\alpha + \beta_1) + \gamma_1, (\alpha_2 + \beta_2) + \gamma_2, \ldots, (\alpha_n + \beta_n) + \gamma_n\} \\
&= \{\alpha + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n\} + w \\
&= (u + v) + w
\end{aligned}
$$

Therefore, V is associative.

iii. *V is distributive*

$$
\begin{aligned}
\lambda \cdot (u + v) &= \lambda\{\alpha + \beta_1, \ \alpha_2 + \beta_2, \ \ldots, \ \alpha_n + \beta_n\} \\
&= \{\lambda(\alpha_1 + \beta_1), \ \lambda(\alpha_2 + \beta_2), \ \ldots, \ \lambda(\alpha_n + \beta_n)\} \\
&= \{\lambda\alpha_1 + \lambda\beta_1, \ \lambda\alpha_2 + \lambda\beta_2, \ \ldots, \ \lambda\alpha_n + \lambda\beta_n\} \\
&= \lambda u + \lambda v
\end{aligned}
$$

$$
\begin{aligned}
\{\lambda + \kappa)u &= \{(\lambda + \kappa)\alpha_1, \ (\lambda + \kappa)\alpha_2, \ \ldots, \ (\lambda + \kappa)\alpha_n\} \\
&= \{\lambda\alpha_1 + \kappa\alpha_1, \ \lambda\alpha_2 + \kappa\alpha_2, \ \ldots, \ \lambda\alpha_n + \kappa\alpha_n\} \\
&= \lambda u + \kappa u
\end{aligned}
$$

Therefore, V is distributive.

iv. *Existence of Negatives*
Every element in u is also an element of R. Therefore, every element in u has an

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additive inverse. The vector in V whose every i^{th} element is the additive inverse of the corresponding element of u is u's additive inverse.

$$\{\alpha_1,\ \alpha_2,\ \ldots\ \alpha_n\} + \{-\alpha_1,\ -\alpha_2,\ \ldots\ -\alpha_n\} = \{\alpha_1 - \alpha_1,\ \alpha_2 - \alpha_2,\ \ldots\ \alpha_n - \alpha_n\}$$
$$= \{0, 0, ..., 0\}$$

Therefore, negatives exist on V

v. *Additive and Multiplicative Identities*
The vector $\{0,\ 0,\ \ldots,\ 0\}$ and the real number 1 are the additive and multiplicative identities in V, respectively.

$$\{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n\} + \{0, 0, ..., 0\} = \{0 + \alpha_1,\ 0 + \alpha_2,\ \ldots,\ 0 + \alpha_n\}$$
$$= \{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n\}$$

$$1 \cdot \{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n\} = \{1 \cdot \alpha_1, 1 \cdot \alpha_2, \ldots,\ 1 \cdot \alpha_n\}$$
$$= \{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n\}$$

Therefore, additive and multiplicative identities exist on V.

vi. *V is closed on vector addition*

$$u + v = \{\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n\}$$

- However, $\alpha_i + \beta_i \in R$ for all $0 < i \le n$.
- This means that $u + v$ is a function from S to R for all $u, v \in V$.
- Therefore, V is closed on vector addition.

vii. *V is closed on scalar multiplication*

$$\lambda \cdot u = \{\lambda\alpha_1, \lambda\alpha_2, \ldots, \lambda\alpha_n\}$$

- However, $\lambda\alpha_i \in R$ for all $0 < i \le n$.
- This means that λu is a function from S to R for all $u, v \in V$.
- Therefore, V is closed on scalar multiplication.

Since these properties hold, V is a vector space on R.

(b) An example of a set of linearly independent generators for V is:

$$B = \left\{ \begin{array}{l} \{1, 0, 0, \ldots, 0\}, \\ \{0, 1, 0, \ldots, 0\}, \\ \{0, 0, 1, \ldots, 0\}, \\ \ldots \\ \{0, 0, 0, \ldots, 1\} \end{array} \right\}$$

where each vector has n coordinates.
- Let $b_1 = \{1, 0, \ldots, 0\}$, $b_2 = \{0, 1,\ \ldots\ , 0\}$, $\ldots$, $b_n = \{0, 0, \ldots, 1\}$
- To prove that this list is a basis, we prove that

i. *B is linearly independent*
We want to show that:
$$\lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_n b_n = 0 \qquad \Longrightarrow \qquad \lambda_1 = \lambda_2 = \ldots = \lambda_n = (0, 0, \ldots, 0)$$

$$
\begin{aligned}
(0, 0, \ldots, 0) \quad &= \quad \lambda_1(1, 0, 0, \ldots, 0) + \lambda_2(0, 1, 0, \ldots, 0) + \ldots + \lambda_n(0, 0, 0, \ldots, 1) \\
&= \quad (\lambda_1, 0, 0, \ldots, 0) \; + \; (0, \lambda_2, 0, \ldots, 0) \; + \ldots + (0, 0, 0, \ldots, \lambda_n) \\
&= \quad (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \\
&\Longrightarrow \quad \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \ldots, \lambda_n = 0.
\end{aligned}
$$

Therefore, the vectors in B are linearly independent

ii. *span(B) = V*
Any vector $v \in V$ of the form $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ can be written by the following linear combination of vectors in B.

$$
\begin{aligned}
(\alpha_1, \alpha_2, \ldots, \alpha_n) \quad &= \quad \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_n b_n \\
&= \quad \alpha_1(1, 0, 0, \ldots, 0) + \alpha_2(0, 1, 0, \ldots, 0) + \ldots + \alpha_n(0, 0, 0, \ldots, 1) \\
&= \quad (\alpha_1, 0, 0, \ldots, 0) \; + \; (0, \alpha_2, 0, \ldots, 0) \; + \ldots + (0, 0, 0, \ldots, \alpha_n) \\
&= \quad (\alpha_1, \alpha_2, \ldots, \alpha_n)
\end{aligned}
$$

- This means that any vector $v \in V$ can be written as a linear combination of the vectors in B.
- Therefore, $span(B) = V$

- Therefore, B is a basis for V.