PROBLEM 5

(a) We want to solve

$$x_1+x_2+x_3=8$$
$$x_1+x_2+x_4=1$$
$$x_1+x_3+x_4=14$$
$$x_2+x_3+x_4=14$$

This yields an augmented Matrix of the following form:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 8 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 14 \\ 0 & 1 & 1 & 1 & 14 \end{array}\right] \underset{row_3 - row_1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 8 \\ 0 & 1 & 1 & 1 & 14 \\ 0 & 0 & -1 & 1 & -7 \\ 1 & 0 & 1 & 1 & 14 \end{array}\right] \underset{row_4 - row_1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 8 \\ 0 & 1 & 1 & 1 & 14 \\ 0 & 1 & -1 & 1 & -7 \\ 0 & -1 & 0 & 1 & 6 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 8 \\ 0 & 1 & 1 & 1 & 14 \\ 0 & 0 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & 20 \end{array}\right] \underset{row_4 + row_2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 8 \\ 0 & 1 & 1 & 1 & 14 \\ 0 & 0 & -1 & 1 & -7 \\ 0 & 0 & 0 & 3 & 13 \end{array}\right]$$

Since our matrix is now upper-triangular, we can translate it back into a system of linear equations,

$$x_1+x_2+x_3=8 \quad (A)$$
$$x_2+x_3+x_4=14 \quad (B)$$
$$-x_3+x_4=-7 \quad (C)$$
$$3x_4=13 \quad (D)$$

Equation (D) yields $x_4=\frac{13}{3}$. Substituting into (C) yields $x_3=\frac{34}{3}$.

Substituting x_3, x_4 into (B) yields $x_2=-\frac{5}{3}$. Similarly, $x_1=\frac{-5}{3}$.

We know that $\left(-\frac{5}{3}, -\frac{5}{3}, \frac{34}{3}, \frac{13}{3}\right)$ is a solution.

Are there more solutions? No. Because, the homogeneous ~~t~~systems of equations only yield $(0,0,0,0)$ at a solution.

(b) $-x_1+x_2+x_4=0$

$x_2+x_3=1$

First, let's find 1 solution. If we let $x_1, x_2, x_4=0$, then $x_3=1$.

$\Rightarrow (0,0,1,0)$ is a solution.

Now, let's solve the homogenous system of equations.

$-x_1+x_2+x_4=0$ If we let $x_1=s$, $x_2=t$.

$x_2+x_3=0$ $\Longrightarrow$ $x_3=-t$, $x_4=s-t$.

$\Rightarrow$ Our solutions can be described as

$$\{(0,0,1,0)+(s,t,-t,s-t)\} = \{(s,t,1-t,s-t) \mid s,t \in \mathbb{R}\}$$

(c) $x_1+4x_2+3x_3=1$

$3x_1+x_3=1$

$4x_1+x_2+2x_3=1$

Our augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 3 & 0 & 1 & 1 \\ 4 & 1 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 3 & 0 & 1 & 1 \\ 4 & 1 & 2 & 1 \end{array}\right] \underset{\substack{\text{row2} \\ -3\text{row1}}}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & -12 & -8 & -2 \\ 4 & 1 & 2 & 1 \end{array}\right] \underset{\substack{\text{row3} \\ -4\text{row1}}}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & -12 & -8 & -2 \\ 0 & -15 & -10 & -3 \end{array}\right]$$

$$\underset{\substack{\text{row3} \\ -\frac{5}{4}\text{row 2}}}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & -12 & -8 & -2 \\ 0 & 0 & 0 & -1/2 \end{array}\right]$$

The bottom row of this matrix implies that

$$0x_1+0x_2+0x_3=-1/2$$

$$\Rightarrow 0=-1/2$$

This is clearly an absurd statement.

$\Rightarrow$ There are no solutions.