PROBLEM SET 9 - PROBLEM 5

(a) To show that $\mathbb{C}^n = U_1 \oplus \cdots \oplus U_s$, we need to show

① $\mathbb{C}^n = U_1 + \cdots + U_s$

② $U_i \cap U_j = \{0\} \quad \forall i \neq j$

① It is obvious that $U_1 + \cdots + U_s \subset \mathbb{C}^n$ because each $U_i \subset \mathbb{C}^n$.
To show the other way, let $z \in \mathbb{C}^n$. So, z can be written as:

$z = a_1 e_1 + \cdots + a_{n_1} e_{n_1} + a_{n_1+1} e_{n_1+1} + \cdots + a_{n_1+n_2} e_{n_1+n_2} + a_{n_1+\cdots n_{s-1}+1} e_{n_1+\cdots+n_{s-1}+1}$
$+ a_n e_n$

But, each $a_{n_1+\cdots n_{i-1}+1} e_{n_1+\cdots+n_{i-1}+1} + \cdots + a_{n_1+\cdots n_i} e_{n_1+\cdots n_i}$
$\in U_i$. Let this element be u_i.

So $z = u_1 + u_2 + \cdots + u_s$

$\Rightarrow \mathbb{C}^n \subset U_1 + \cdots + U_s$.

② Let $v \in U_i \cap U_j \quad i \neq j$

$\Rightarrow v \in U_i$ and $v \in U_j$

So, we can write v as a linear combination of the vectors that span U_i and also as a different linear combination of the vectors that span U_j.

So. $v = a_{n_1+\cdots+n_{i-1}+1} e_{n_1+\cdots+n_{i-1}+1} + \cdots + a_{n_1+\cdots+n_i} e_{n_1+\cdots+n_i} \} U_i$

$v = a_{n_1+\cdots+n_{j-1}+1} e_{n_1+\cdots+n_{j-1}+1} + \cdots + a_{n_1+\cdots+n_j} e_{n_1+\cdots+n_j}$

Subtracting these two equations yields

$$0 = a_{n_1+\cdots+n_{i-1}+1} e_{n_1+\cdots+n_{i-1}+1} + \cdots + a_{n_1+\cdots+n_i} e_{n_1+\cdots+n_i}$$

$$- a_{n_1+\cdots+n_{j-1}+1} e_{n_1+\cdots+n_{j-1}+1} - \cdots - a_{n_1+\cdots+n_j} e_{n_1+\cdots+n_j}$$

But, since $e_1, \ldots, e_n$ form a basis, they are linearly independent.

So, all of the coefficients are 0. Therefore, $v = 0$.

(c) First, we will show that $U_i \in \ker(A-\lambda_i \mathbb{1})^N$

$$(A-\lambda_i\mathbb{1})^N = \begin{bmatrix} (A-\lambda_i\mathbb{1})^N & & & \\ & \ddots & & 0 \\ & & (A_i-\lambda_i\mathbb{1})^N & \\ & 0 & & \ddots \\ & & & (A_s-\lambda_i\mathbb{1})^N \end{bmatrix}$$

But, $A_i-\lambda_i\mathbb{1}) = \begin{bmatrix} \lambda_i-\lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i-\lambda_i \end{bmatrix} = \begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$

Therefore, by 4c), we have

$$(A_i-\lambda_i\mathbb{1})^N = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{if } N \geq n_i$$

So,

$$(A-\lambda_i\mathbb{1})^N = \begin{bmatrix} \begin{bmatrix} \lambda_1-\lambda_i & & * \\ 0 & \ddots & \lambda_1-\lambda_i \end{bmatrix} & & & 0 \\ & \ddots & & \\ & & \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix} & \\ 0 & & & \ddots \\ & & & \begin{bmatrix} \lambda_s-\lambda_i & & * \\ 0 & \ddots & \lambda_s-\lambda_i \end{bmatrix} \end{bmatrix}$$

So, the columns that contain the i^{th} block (columns $n_1+\cdots+n_{i-1}+1$ through $n_1+\cdots+n_i$) contain zeroes.

So, $(A-\lambda_i\mathbb{1})^N e_{n_1+\cdots n_{i-1}+1} = 0$

$\vdots$

$(A-\lambda_i\mathbb{1})^N e_{n_1+\cdots+n_i} = 0$

If $v_i \in U_i$, we can write it as

$$v_i = a_{n_1+\cdots+n_{i-1}+1}\, e_{n_1+\cdots+n_{i-1}+1} + \cdots + a_{n_1+\cdots+n_i}\, e_{n_1+\cdots+n_i}$$

So, $(A-\lambda_i \mathbb{1})^N u_i =$

$$(A-\lambda_i \mathbb{1})\left[a_{n_1+\cdots+n_{i-1}+1} e_{n_1+\cdots+n_{i-1}+1} + \cdots + a_{n_1+\cdots+n_i} e_{n_1+\cdots+n_i} \right]$$

But, $(A-\lambda_i \mathbb{1})$ applied to and yield 0.

So, $(A-\lambda_i \mathbb{1}) U_i = 0$

Therefore $u_i \in \ker(A-\lambda_i \mathbb{1})^N$.

Now, let $v \in \ker(A-\lambda_i \mathbb{1})^N$.

$$\Rightarrow (A-\lambda_i \mathbb{1})^N v = 0$$

$$\Rightarrow \begin{bmatrix} \begin{matrix} \lambda_1-\lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_1-\lambda_i \end{matrix} & & \\ & \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \\ & & \begin{matrix} \lambda_s-\lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_s-\lambda_1 \end{matrix} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{n_1} \\ \vdots \\ v_{n_1+\cdots+n_{i-1}+1} \\ \vdots \\ v_{n_1+\cdots+v_{n_i}} \\ v_{n_1+\cdots+n_{s-1}+1} \\ \vdots \\ v_n \end{bmatrix} = 0$$

This yields:

$$\begin{bmatrix} \lambda_1-\lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_1-\lambda_i \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{n_1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_{n_1+\cdots n_{i-1}+1} \\ \vdots \\ v_{n_1+\cdots n_i} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_s-\lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_s-\lambda_1 \end{bmatrix} \begin{bmatrix} v_{n_1+\cdots+n_{s-1}+1} \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

For $j \neq i$

$$\begin{bmatrix} \lambda_j - \lambda_i & & * \\ & \ddots & \\ & & \lambda_j - \lambda_i \end{bmatrix} \begin{bmatrix} v_{n_1+\cdots+n_{j-1}+1} \\ \vdots \\ v_{n_1+\cdots+n_j} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$$\Rightarrow (\lambda_j - \lambda_i)(v_{n_1+\cdots n_j}) = 0$$

$$\Rightarrow v_{n_1+\cdots+n_j} = 0$$

$$(\lambda_j - \lambda_i)(v_{n_1+\cdots+n_j-1}) + \underset{\substack{\uparrow \\ \text{Some entry} \\ \text{we don't care about.}}}{*}(v_{n_1+\cdots+n_j}) = 0$$

$$\Rightarrow v_{n_1+\cdots+n_j-1} = 0$$

By the same reasoning, we know all the entries of v corresponding to blocks where $j \neq i$ must be 0.

The only entries that don't need to be 0 are those where $j = i$.

So, $v = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ v_{n_1+\cdots+n_{i-1}+1} \\ \vdots \\ v_{n_1+\cdots+n_i} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$v = v_{n_1+\cdots+n_{i-1}+1} e_{n_1+\cdots+n_{i-1}+1} + \cdots + v_{n_1+\cdots n_i} e_{n_1+\cdots+n_i}$$

$$\Rightarrow v \in U_i$$