Math 23aTheoretical Linear Algebra and Multivariable Calculus I

PROBLEM SET 1

Problem 1: Consider the following four sets:

$$A = \{1,2\}\ ,\quad B = \{\{1\},\{2\}\}\ ,$$
$$C = \{\{1\},\{1,2\}\}\ ,\quad D = \{\{1\},\{2\},\{1,2\}\}\ .$$

For each of the following statements, argue whether it is true or false:
(1) $A = B$
(2) $A \subset C$
(3) $A \in C$
(4) $A \subset D$
(5) $B \subset C$
(6) $B \subset D$
(7) $B \in D$
(8) $A \in D$

Problem 2: Prove the following relations
(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(2) $A \cup (A \cap B) = A = A \cap (A \cup B)$
(3) $A - (B \cap C) = (A - B) \cup (A - C)$
(4) If $\mathcal{F}$ is a class of sets, $B - \bigcup_{A\in\mathcal{F}} A = \bigcap_{A\in\mathcal{F}}(B - A)$

Problem 3: In the following exercise we will construct the field of rational numbers $\mathbb{Q}$ starting from the set of integers $\mathbb{Z}$.

I will assume we all know and understand what are integer numbers, and what it means to add or multiply two integers. (But we do NOT know how to divide two integers!)

(1) Consider the set $\mathbb{Z} \times \mathbb{Z}^*$, where $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$. Define the following relation $\sim$ on this set

$$(a,b) \sim (c,d) \quad \Longleftrightarrow \quad ad = bc\ .$$

Prove that $\sim$ is an equivalence relation.

(2) We can then take the associated partition, namely the set of equivalence classes, that we denote as follows

$$\mathbb{Q} \ =\ \mathbb{Z}\times\mathbb{Z}^*/\sim\ =\ \left\{[(a,b)]\ \middle|\ a\in\mathbb{Z},\ b\in\mathbb{Z}^*\right\}\ .$$

We define the following operations of addition ($+_{\mathbb{Q}}$) and multiplication ($\cdot_{\mathbb{Q}}$) on $\mathbb{Q}$:

$$[(a,b)]\ +_{\mathbb{Q}}\ [(c,d)] \ =\ [(ad+bc, bd)]\ ,$$
$$[(a,b)]\ \cdot_{\mathbb{Q}}\ [(c,d)] \ =\ [(ac, bd)]\ .$$

Prove that these operations are well defined on $\mathbb{Q}$.

(3) Prove that $\mathbb{Q}$, together with the above operations of addition and multiplication, is a field. In particular, which are the identity elements, what is the negative of an element $[(a,b)]$, and what is its inverse?

(**Hint:** Think at the pair (a, b) as the "number" $\frac{a}{b}$).

Problem 4: Let F be a field. Prove the following statements (using only the field axioms and the results we proved in class)

(1) $a \cdot (b - c) = a \cdot b - a \cdot c$
(2) $0 \cdot a = 0$
(3) If $ab = ac$ and $a \neq 0$, then $b = c$
(4) Given $a, b \in F$ such that $a \neq 0$, there is exactly one $x \in F$ such that $ax = b$ (and from now on we will denote it by $x = b/a$
(5) If $a \neq 0$, then $(a^{-1})^{-1} = a$
(6) If $a \cdot b = 0$, then $a = 0$ or $b = 0$
(7) $(-a) \cdot b = -(a \cdot b) = a \cdot (-b)$
(8) If $b, d \neq 0$, we have $a/b + c/d = (a \cdot d + b \cdot c)/(b \cdot d)$

Problem 5: Let F be a field. Prove the following statements

(1) $-0 = 0$
(2) $1^{-1} = 1$
(3) $-(a + b) = -a - b$
(4) $(a - b) + (b - c) = a - c$
(5) If $a, b \neq 0$, then $(a \cdot b)^{-1} = a^{-1}b^{-1}$

Problem 6: Check if this problem set was:

- Too hard
- Too easy
- just about right

Comments: