MATH 23A P-SET 10

Question 1

(a) For this we can just use properties of matrix multiplication.

Consider w, x, y, z column vectors in $\mathbb{R}^n$, $\lambda \in \mathbb{R}$

$$\langle w+x | y\rangle = (w+x)^T A y = (w^T + x^T) A y = w^T A y + x^T A y$$
$$= \langle w|y\rangle + \langle x|y\rangle$$

$$\langle \lambda x | y\rangle = (\lambda x)^T A y = \lambda (x^T A y) = \lambda \langle x|y\rangle$$

$$\langle x | y+z\rangle = x^T A (y+z) = x^T A y + x^T A z = \langle x|y\rangle + \langle x|z\rangle$$

$$\langle x | \lambda y\rangle = x^T A (\lambda y) = \lambda (x^T A y) = \lambda \langle x|y\rangle$$

(b) Consider $e_j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \leftarrow$ jth entry , $\{e_1, \dots, e_n\}$ forms a basis for $\mathbb{R}^n$

$\langle \cdot | \cdot \rangle$ is symmetric $\Rightarrow \langle e_i, e_j\rangle = \langle e_j, e_i\rangle$

$\leftarrow c_j$ is the j-th column of A

But $\langle e_i, e_j \rangle = e_i^T A e_j = e_i^T c_j = a_{ij}$

$\langle e_j, e_i\rangle = e_j^T A e_i = a_{ji}$ So $a_{ij} = a_{ji} \Rightarrow A$ is Symmetric.

This is necessary. It is also sufficient, because if A is symmetric, then $u^T A v = u^T A^T v = (v^T A u)^T = v^T A u$

Note $x^T A y = (x^T A y)^T$ $\forall x, y$, because $x^T A y$ is a 1×1 matrix.

(c) By (b) we know that A is of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

We have symmetry & bilinearity, so we must now find conditions to make $\langle \cdot | \cdot \rangle$ positive definite

For $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\langle x|x\rangle > 0 \Rightarrow [1\ 0]\begin{bmatrix} a \\ b \end{bmatrix} > 0 \Rightarrow a > 0$

For $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\langle x|x\rangle > 0 \Rightarrow c > 0$

If $x = \begin{pmatrix} 1/\sqrt{a} \\ -1/\sqrt{c} \end{pmatrix}$ $\langle x|x\rangle > 0 \Rightarrow 2 > \frac{2b}{\sqrt{ac}} \Rightarrow b < \sqrt{ac}$

If $x = \begin{pmatrix} 1/\sqrt{a} \\ 1/\sqrt{c} \end{pmatrix}$ $\langle x|x\rangle > 0 \Rightarrow b > -\sqrt{ac}$

So our necessary conditions are that

$a > 0, \quad c > 0, \quad -\sqrt{ac} < b < \sqrt{ac}$.

To check sufficiency, note that we want to prove

$x^T A x > 0$ for arbitrary $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

This is the same as

$$a x_1^2 + 2b x_1 x_2 + c x_2^2 > 0$$

Now, consider the conditions above.

And consider $x_1 \cdot x_2 \geq 0$

If these hold, then

$$a x_1^2 + 2b x_1 x_2 + c x_2^2 > a x_1^2 - 2\sqrt{ac}\, x_1 x_2 + c x_2^2 = (\sqrt{a}\, x_1 - \sqrt{c}\, x_2)^2 \geq 0$$

If $x_1 \cdot x_2 \leq 0$

$$\text{then } a x_1^2 + 2b x_1 x_2 + c x_2^2 > a x_1^2 + 2\sqrt{ac}\, x_1 x_2 + c x_2^2 = (\sqrt{a}\, x_1 + \sqrt{c}\, x_2)^2 \geq 0$$

So they are sufficient.

Thus A produces an inner product iff

$a > 0, \quad c > 0, \quad$ and $\quad ac > b^2$ (this is the same as $-\sqrt{ac} < b < \sqrt{ac}$)