Math 23aTheoretical Linear Algebra and Multivariable Calculus I

PROBLEM SET 10

Problem 1: Let A be a $n \times n$ matrix with real coefficients. Consider the map $\langle \cdot | \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = [x_1, \cdots, x_n] A \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

(On the right hand side we perform row by column mulitplication).

(a) Prove that $\langle \cdot | \cdot \rangle$ is bilinear (i.e. linear in first and second factor).

(b) Find the necessary and sufficient condition on the matrix A that makes $\langle \cdot | \cdot \rangle$ symmetric.

(c) Let $n = 2$. Find necessary and sufficient conditions on A that make $\langle \cdot | \cdot \rangle$ an inner product.

Problem 2: For the following real vector spaces V determine which of the following maps $\langle \cdot | \cdot \rangle : V \times V \to \mathbb{R}$ is an inner product on V. In case $\langle \cdot | \cdot \rangle$ is not an inner product, tell which axioms are not satisfied.

(a) $V = \mathbb{R}^n$, $\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \sum_{i=1}^n x_i |y_i|.$

(b) $V = \mathbb{R}^n$, $\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = (\sum_{i=1}^n x_i^2 y_i^2)^{1/2}.$

(c) $V = \mathbb{R}^n$, $\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \sum_{i=1}^n (x_i + y_i)^2 - \sum_{i=1}^n x_i^2 - \sum_{i=1}^n y_i^2.$

(d) V is the space of all real polynomials in x, and $\langle P(x) | Q(x) \rangle = P(1)Q(1)$,

(e) V is the space of all real polynomials in x, and $\langle P(x) | Q(x) \rangle = \int_0^1 P'(x) Q'(x) dx$, where $P'(x)$ denotes the derivative of $P(x)$.

Problem 3: Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$. Prove that

$$\left(\sum_{j=1}^n a_j b_j \right)^2 \leq \left(\sum_{j=1}^n j a_j^2 \right) \left(\sum_{j=1}^n b_j^2 / j \right)$$

Problem 4: Consider the space $\mathcal{P}$ of polynomials in x with real coefficients, and let $\langle \cdot | \cdot \rangle$ be the following inner product:

$$\langle P(x) | Q(x) \rangle = \int_0^1 P(x) Q(x) dx$$

Let $V \subset \mathcal{P}$ be the subspace generated by the following elements

$$V = \operatorname{span}\{1, 1 + x, x^2, x^4\}.$$

Use the Gram-Schmidt procedure to find a basis of V orthonormal with respect to $\langle \cdot \,|\, \cdot \rangle$.

Problem 5: A *norm* on a vector space V (over $\mathbb{R}$) is a map $|| \cdot || : V \to \mathbb{R}$ satisfying the following axioms:

i. $||cv|| = |c| \, ||v||$, $\forall c \in \mathbb{R}$, $v \in V$,

ii. $||v|| \geq 0$, $\forall v \in V$, and $||v|| = 0$ if and only if $v = 0$,

iii. $||u + v|| \leq ||u|| + ||v||$, $\forall u, v \in V$.

Prove that

(a) Prove that, if V is a Euclidean space, then $||v|| = \sqrt{\langle v|v \rangle}$ is a norm on V.

(b) Consider the map $|| \cdot || : \mathbb{R}^n \to \mathbb{R}$ given by

$$\left|\left| \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right|\right| = \max \{|x_1|, \cdots, |x_n|\}$$

(i.e. the maximum number among $|x_1|, \cdots, |x_n|$) defines a norm on $\mathbb{R}^n$.

(c) Prove that there is no inner product $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{R}^n$ such that $||v|| = \sqrt{\langle v|v \rangle}$ (where $|| \cdot ||$ is as in part (b)). (**Hint:** if $\langle \cdot \,|\, \cdot \rangle$ is an inner product, what is $\langle v + u|v + u \rangle + \langle v - u|v - u \rangle$?)