

Preface: Common Errors We'll have this section at the beginning of the solution sets to talk about common errors that came up, and other tips we can give you. We'll add to it as we think of more stuff to say. For now:

- Ideally, people should prove things by changing only one side, or being careful about what they do to both sides. It's generally bad to prove by starting with the statement and simplifying it to something true (e.g. $1 = 2$ because $1 * 0 = 2 * 0$). This is bad because it only works if all your claims are true in the other direction, (which isn't the case in the example), and because, at best, it is an indirect and less enlightening approach. It can be helpful for telling you where to start, but write your final proof more directly.
- Be careful when making assumptions about collections of objects, whether sets, vector spaces, etc. Unless specifically stated in the problem, we do not know whether they are finite, countably infinite, uncountably infinite, or even bigger...

That's it for now. Check back later, we might have added more. Everyone keep up the good work: it will only get less tedious and more interesting from here. If you have questions about the graded sets, you can ask us during office hours.

Problem 1

We have the following four sets:

$$A = \{1, 2\}, B = \{\{1, \}, \{2\}\}, C = \{\{1\}, \{1, 2\}\}, D = \{\{1\}, \{2\}, \{1, 2\}\}$$

(1) $A = B$

False, since $1 \in A$ but not B .

(2) $A \subset B$

False, since $1 \in A$ but not C .

(3) $A \in C$

True, since the set $\{1, 2\}$ is one of the members of C .

(4) $A \subset D$

False, since $1 \in A$ but not D .

(5) $B \subset C$

False, since $\{2\} \in B$ but not C .

(6) $B \subset D$

True, since both of B 's elements, $\{1\}$ and $\{2\}$, are elements of D .

(7) $B \in D$

False, since $\{\{1, \}, \{2\}\}$ is not a member of D .

(8) $A \in D$

True, since $\{1, 2\}$ is an element of D .

Problem 2

(1) Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $a \in A \cup (B \cap C)$. Then by the definition of the union $a \in A$ or $a \in B \cap C$. If $a \in A$, then $a \in A \cup B$ and $a \in A \cup C$, so $a \in (A \cup B) \cap (A \cup C)$. If $a \in B \cap C$, $a \in B$ and $a \in C$, so $a \in A \cup B$ and $a \in A \cup C$, and so $a \in (A \cup B) \cap (A \cup C)$. Therefore, $a \in (A \cup B) \cap (A \cup C)$. Since we chose a arbitrarily, we can conclude that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. Now, let $b \in (A \cup B) \cap (A \cup C)$. If $b \in A$, then $b \in A \cup (B \cap C)$. If b is not in A , then since $b \in A \cup B$ and $b \in A \cup C$, we can conclude that $b \in B$ and $b \in C$. So $b \in B \cap C$, so $b \in A \cup (B \cap C)$. Therefore, regardless of whether $b \in A$, $b \in A \cup (B \cap C)$. Since b was chosen arbitrarily, $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$. Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(2) Show that $A \cup (A \cap B) = A = A \cap (A \cup B)$

If $a \in A$, then $A \in A \cup (A \cap B)$, so $A \subset A \cup (A \cap B)$. If $a \in A \cup (A \cap B)$, then $a \in A$ or $a \in A \cap B$. Since $a \in A \cap B \implies a \in A$, either way $a \in A$. Therefore $A \cup (A \cap B) \subset A$, so $A \cup (A \cap B) = A$.

To show that $A \cup (A \cap B) = A \cap (A \cup B)$, we apply Problem 2 (1) with $C = A$; since $A \cup (B \cap A) = (A \cup A) \cap (A \cup B)$, and $A \cup A = A$, we can conclude that $A \cup (A \cap B) = A \cap (A \cup B)$.

Therefore $A \cup (A \cap B) = A = A \cap (A \cup B)$.

(3) Show that $A - (B \cap C) = (A - B) \cup (A - C)$.

Let $a \in A - (B \cap C)$.

(4) If F is a class of sets, show that $B - \bigcup A = \bigcap (B - A)$ over all $A \in F$.

Let $a \in B - \bigcup A$. Then $a \in B$ but a is not in any set $A \in F$. Therefore a is in every $B - A$ with $A \in F$, so $a \in \bigcap (B - A)$, so $B - \bigcup A \subset \bigcap (B - A)$.

Let $a \in \bigcap (B - A)$. Then $a \in B - A$ for all $A \in F$, $a \in B$ but a is not in A for any $A \in F$. Therefore, a is not in $\bigcup A$, but since $a \in B$, $a \in B - \bigcup A$. Therefore, $\bigcap (B - A) \subset B - \bigcup A$.

Therefore, $B - \cup A = \cap(B - A)$.

Problem 3

Show that \sim is an equivalence relation.

To be an equivalence relation, \sim must be reflexive, symmetric, and transitive.

Reflexive property:

$(a, b) \sim (a, b) \iff ab = ba$, which is true by the commutative property of multiplication.

Symmetric property:

Assume $(a, b) \sim (c, d)$, so $ad = cb$. Then by commutativity, $da = cb$, so $(c, d) \sim (b, a)$.

Transitive property:

Assume $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $ad = bc$ and $cf = de$. Multiplying these together gives:

$$adcf = bcde$$

$$af(cd) = be(cd) \text{ (by commutativity)}$$

$$af(cd)(cd)^{-1} = be(cd)(cd)^{-1} \text{ if } cd \neq 0$$

$$af = be \implies (a, b) \sim (e, f)$$

However, if $cd = 0$, since $d \neq 0$ we know $c = 0$ (we will prove this in problem 4). So $ad = bc$ and $b, d \neq 0$ imply that $a = 0$, and likewise $cf = de$ and $d, f \neq 0$ imply that $e = 0$. Therefore $af = a*0 = 0$ and $be = b*0 = 0$, again using results from problem 4. So, $af = be$, and $(a, b) \sim (e, f)$. Therefore $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ imply $(a, b) \sim (e, f)$.

Since \sim is symmetric, transitive, and reflexive, it is an equivalence relation.

(2) Show that $+_Q$ and $*_Q$ are well defined.

For $+_Q$ to be well defined, it must always give the same result when applied to different members of the same equivalence classes. By symmetry and commutativity, it is sufficient to show that if $(a, b) \sim (e, f)$, then $[(a, b)] +_Q [(c, d)] = [(e, f)] +_Q [(c, d)]$.

Note: What follows is one possible way to go about the proof. It is certainly possible to prove the result without introducing this lemma (which confirms our intuitive view of equivalence classes), and to prove everything directly from the definition of equivalence class.

First, we introduce the lemma that for all $a, b, c \in \mathcal{Z}$ with $c \neq 0$, $(a, b) \sim (ac, bc)$. This is true since $a(bc) = b(ac)$ by associativity and commutivity of multiplication.

Assume that $(a, b) \sim (e, f)$, so $af = be$, and $adf = bde$.

$$[(a, b)] +_Q [(c, d)] = [(ad + bc, bd)]$$

$$[(a, b)] +_Q [(c, d)] = [((ad + bc)f, bdf)] \text{ (see above lemma).}$$

$$[(a, b)] +_Q [(c, d)] = [(adf + bcf, bdf)]$$

$$[(a, b)] +_Q [(c, d)] = [(bde + bcf, bdf)] \text{ (by substitution from above)}$$

$$[(a, b)] +_Q [(c, d)] = [(de + cf, df)] \text{ (factoring out } b \text{ and applying the lemma)}$$

$$[(a, b)] +_Q [(c, d)] = [(e, f)] +_Q [(c, d)] \text{ (by the definition of } +_Q)$$

Therefore $+_Q$ is well-defined.

Likewise, to show that $*_Q$ is well defined, we must show that $[(a, b)] *_Q [(c, d)] = [(e, f)] *_Q [(c, d)]$. Again assume $(a, b) \sim (e, f)$, so $af = be$.

$$[(a, b)] *_Q [(c, d)] = [(ac, bd)]$$

$$[(a, b)] *_Q [(c, d)] = [(acaf, bda f)] \text{ applying the lemma}$$

$$[(a, b)] *_Q [(c, d)] = [(abce, abdf)] \text{ substituting from above}$$

$$[(a, b)] *_Q [(c, d)] = [(ce, df)] \text{ again applying the lemma}$$

$$[(a, b)] *_Q [(c, d)] = [(c, f)] *_Q [(c, d)]$$

Therefore, $*_Q$ is well-defined.

(3) The closure of multiplication and addition follows directly from the closure of the integers under the same operations, and the theorem that $b, d \neq 0 \Rightarrow bd \neq 0$. Commutativity, as mentioned before, also follows directly. Associativity and distributivity are just a matter of expanding the various combinations of operations, and demonstrating that equality follows from properties of multiplication and addition on \mathbb{Z} . For example, I will show associativity of addition: We want to show that

$$([(a, b)] +_Q [(c, d)]) +_Q [(e, f)] = [(a, b)] +_Q (([c, d)] +_Q [(e, f)]) \Leftrightarrow$$

$$[(ad + bc, bd)] + [(e, f)] = [(a, b)] + [(cf + de, df)] \Leftrightarrow$$

$$[(ad + bc)f + bde, bdf] = [(adf + b(cf + de), bdf)] \Leftrightarrow$$

$$[(adf + bcf + bde, bdf)] = [(adf + bcf + bde, bdf)].$$

It is also direct to show, once we know that the operations are well-defined, that $[(0, 1)]$ is an identity element w.r.t addition, $[(1, 1)]$ is the identity w.r.t. multiplication, and that $[(-a, b)]$ and $[(b, a)]$ are the respective inverses of $[(a, b)]$. (I'm omitting this stuff from the solution set because the other parts are similar. However, you still had to do all the parts on your problem sets) :-)

Problem 4—Coming soon!

Problem 5

1. $-0 + 0 = 0$ by additive inverse
 $-0 + 0 = -0$ by additive identity

$$\Rightarrow -0 = 0.$$

2. $1 * 1^{-1} = 1$ by multiplicative inverse

$$1 * 1^{-1} = 1^{-1} \text{ by multiplicative identity}$$

$$\Rightarrow 1^{-1} = 1$$

3. $0 = 0 + 0$ by additive identity

$$(a + b) + (-(a + b)) = a + (-a) + b + (-b) \text{ by the definition of inverses (thrice)}$$

$$(a + b) + (-(a + b)) = (a + b) + (-a) + (-b) \text{ by associativity}$$

$$-(a + b) = -a + (-b) \text{ by cancellation law}$$

$$-(a + b) = (-a - b) \text{ by the definition of subtraction}$$

4. $(a - b) + (b - c) = (a + (-b)) + (b + (-c))$ by the definition of subtraction

$$\text{RHS} = (a + ((-b) + b) + (-c) \text{ by associativity}$$

$$\text{RHS} = a + (-c) \text{ by additive inverse}$$

$$\text{RHS} = a - c \text{ by the definition of subtraction}$$

5. $1 * 1 = 1$ by multiplicative identity

$$(a * a^{-1})(b * b^{-1}) = 1 \text{ by the definition of multiplicative inverse}$$

$$(ab)(a^{-1}b^{-1}) = 1 \text{ by associativity}$$

$$\Rightarrow ab \text{ and } a^{-1}b^{-1} \text{ multiply to get 1 and so are multiplicative inverses, i.e. } (ab)^{-1} = a^{-1}b^{-1}.$$