Math 23aTheoretical Linear Algebra and Multivariable Calculus I

PROBLEM SET 3

Problem 1: (a) Let V be a given vector space over a certain field $\mathbb{F}$. Prove that $(-1) \cdot v \ = \ -v$

(b) Prove that the subset $U \subset \mathbb{R}^3$ consisting of all 3-columns $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ such that $2x + y \ = \ 0$ and $3y - z \ = \ 0$ is in fact a subspace of $\mathbb{R}^3$.

(c) Show that if $\dim V \ = \ n$, than any list of $n + 1$ vectors is linearly dependent.

Problem 2: Consider the field $\mathbb{F}_7 = \mathbb{Z}/7\mathbb{Z}$. For simplicity, I will denote the elements of this field as $0, 1, 2, \dots$, instead of $[0], [1], [2], \dots$ (hoping not to create confusion). Consider the vector space $V = (\mathbb{F}_7)^2$.

(a) How many distinct vectors are in the vector space V?

(b) Consider the vectors $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Prove that

- (a) they generate the whole vector space V,
- (b) they are linearly independent.

(c) Consider the vectors $a_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $a_3 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Prove that

- (a) they generate the whole vector space V,
- (b) they are linearly dependent.

(d) Find an explicit example, writing down all elements, of a proper (i.e. neither $\{0\}$ nor V) subspace of V.

(e) How many distinct subspaces does V have?

Problem 3: Consider the vector space $C[0, 1]$ of all continuous functions $f : [0, 1] \to \mathbb{R}$. (Here $[0, 1]$ denotes the set of all real numbers x such that $0 \le x \le 1$.) For each $c \in \mathbb{R}$, consider the following subset:

$$V(c) \ = \ \big\{ f \in C[0, 1] \mid f(1/2) = c \big\} \subset C[0, 1] \ .$$

Find all values of c for which $V(c)$ is a subspace of $C[0, 1]$ (Explain your answer).

Problem 4: Let $S = \{1, 2, \dots, n\}$ for some positive integer n, and consider the set V of all functions $f : \ S \to \mathbb{R}$.

(a) Prove that V is a vector space over $\mathbb{R}$. (What are the vector space addition and scalar multiplication? And what is the zero vector?)

(b) Find a linearly independent list of generators of V (i.e. a basis).

Problem 5: Let V be the set of all infinite sequences of elements of $\mathbb{R}$:

$$V \ = \ \big\{ (\alpha_0, \alpha_1, \alpha_2, \dots) \mid \alpha_i \in \mathbb{R} \ \forall i \ge 0 \big\} \ .$$

(a) Prove that V is a vector space over $\mathbb{R}$ (what are the vector space addition and scalar multiplication?)

Consider now the following subsets of V:

$$\Lambda_1 = \left\{ (\alpha_0, \alpha_1, \alpha_2, \dots) \mid \alpha_i \in \mathbb{R} \ \forall i \geq 0 \text{ and } \sum_{n=0}^{\infty} |a_n| \text{ converges} \right\},$$

$$\Lambda_2 = \left\{ (\alpha_0, \alpha_1, \alpha_2, \dots) \mid \alpha_i \in \mathbb{R} \ \forall i \geq 0 \text{ and } \sum_{n=0}^{\infty} |a_n|^2 \text{ converges} \right\},$$

(b) Prove that Λ_1 and Λ_2 are subspaces of V
(c) Prove that $\Lambda_1 \subset \Lambda_2$ but $\Lambda_1 \neq \Lambda_2$.

(**Note:** you use appropriate results from Calculus).

We define an *arithmetic sequence* to be a sequence $(\alpha_0, \alpha_1, \alpha_2, \dots) \in V$ for which there exists some constant $c \in \mathbb{R}$ such that $\alpha_{n+1} = \alpha_n + c, \ \forall n \geq 0$. Consider the subset $W \subset V$ consisting of all arithmetic sequences.

(d) Prove that W is a subspace of V.
(e) Find a linearly independent list of generators of W (i.e. a basis). What is the dimension of W?