# MATH 23A, P-SET 4 Question #1

$W$ is a vector space. $U$ & $V$ are subspaces.

$U+V = \{u+v \in W \mid u \in U, v \in V\}$

(a) We must check that $U+V$ is

- non-empty. But $0 \in U, 0 \in V \Rightarrow 0+0 = 0 \in U+V$
- closed under addition. So for $u_1+v_1, u_2+v_2 \in U+V$

$$(u_1+v_1)+(u_2+v_2) = \underset{\in U}{(u_1+u_2)} + \underset{\in V}{(v_1+v_2)} \in U+V$$

- closed under scalar mult.

for $u+v \in U+V$ $\lambda \in \mathbb{F}$

$$\lambda(u+v) = \underset{\in U}{(\lambda u)} + \underset{\in V}{(\lambda v)} \in U+V$$

So $U+V$ is a subspace.

(b) Let $T$ be an arbitrary subspace containing $U \cup V$. $T$ is closed under addition, so

$\forall u \in U, \forall v \in V \quad u+v \in T$. But these are precisely the elements of $U+V \Rightarrow U+V \subset T$.

Now we note that for $u \in U$ $\quad u = \underset{\in U}{u} + \underset{\in V}{0} \in U+V$

and for $v \in V$ $\quad v = \underset{\in U}{0} + \underset{\in V}{v} \in U+V$

$\Rightarrow \quad U \subset U+V \;\&\; V \subset U+V \Rightarrow U \cup V \subset U+V$

So $U+V$ contains $U \cup V$ and any subspace containing $U \cup V$ contains $U+V$ implies that $U+V$ is the smallest subspace containing $U \cup V$ (we know that $U+V$ is a subspace from (a)).

(c) Remember from the midterm that $U \cap V$ is a subspace (unless $U \cap V = \{0\}$).

First let's consider the case $U \cap V = \{0\}$

If $\dim(U) = n$, $\dim(V) = m$, we can take bases $\{u_1, \ldots, u_n\}$ & $\{v_1, \ldots, v_m\}$ respectively.

Note that $\{u_1, \ldots, u_n, v_1, \ldots, v_m\}$ certainly spans $U \cup V$, thus it must span $U+V$ by (b). And this set must be linearly independent, otherwise $\exists$ a linear combination $\alpha_i, \lambda_i \neq 0 \; \forall i$

$$\underbrace{\lambda_1 u_1 + \cdots + \lambda_n u_n}_{\in U} = \underbrace{-\alpha_1 v_1 - \cdots - \alpha_m v_m}_{\in V}$$

$$\Rightarrow \quad U \cap V \neq \{0\}$$

So this set is a basis $\Rightarrow \dim U+V = n+m = \dim U + \dim V - \underbrace{\dim U \cap V}_{0}$

Now consider $U \cap V \neq \{0\}$

Then $U \cap V$ is a subspace. So take $\{a_1, \ldots, a_k\}$ a basis (where

$k = \dim U \cap V)$

This set is linearly independent in both $U$ & $V$ $\Rightarrow$ it can be extended to ~~a~~ bases $\{a_1, \ldots, a_k, u_1, \ldots, u_{n-k}\}$ and $\{a_1, \ldots, a_k, v_1, \ldots, v_{m-k}\}$ of $U$ & $V$ respectively.

I claim that $\{a_1, \ldots, a_k, u_1, \ldots, u_{n-k}, v_1, \ldots, v_{m-k}\}$ is a basis for $U+V$.

This ~~set~~ contains bases for $U$ & $V$ so it certains spans $U \cup V$, thus ~~it~~ spans $U+V$. And if the set was linearly dependent, then $\exists \lambda_i, \alpha_j, \beta_k \neq \forall i,j,k$ such that

$$\underbrace{\lambda_1 a_1 + \cdots + \lambda_k a_k + \alpha_1 u_1 + \cdots \alpha_{n-k} u_{n-k}}_{\in U} = \underbrace{-\beta_1 v_1 - \cdots - \beta_{m-k} v_{m-k}}_{\in V - U \cap V}$$

But then this implies that $-\beta_1 v_1 - \cdots - \beta_{m-k} v_{m-k} \in U$ & is in $V$

$\Rightarrow$ $-\beta_1 v_1 - \cdots - \beta_{m-k} v_{m-k} \in U \cap V$

This contradicts the fact that $\{a_1, \dots, a_k\}$ is a basis for $U \cap V$ and $\{a_1, \dots, a_k, v_1, \dots, v_{m-k}\}$ is linearly independent.

$\Rightarrow \beta_k = 0 \quad \forall k$. But $\{a_1, \dots, a_k, u_1, \dots, u_{n-k}\}$ is linearly independent

$\Rightarrow \lambda_i = \alpha_j = 0 \quad \forall i, j$

So $\{a_1, \dots, a_k, u_1, \dots, u_{n-k}, v_1, \dots, v_{m-k}\}$ spans $U+V$ and is linearly independent $\Rightarrow$ it is a basis.

So $\dim(U+V) = (n-k) + (m-k) + k = n + m - k = \dim U + \dim V - \dim U \cap V$

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