Math 23aTheoretical Linear Algebra and Multivariable Calculus I

PROBLEM SET 4

Problem 1: Suppose W is a vector space over some field $\mathbb{F}$, and let U, V be two subspaces of W. In the exam you (were supposed to) prove that $U \cap V$ is again a subspace of W, while in general $U \cup V$ is not a subspace of W. We want to define a new subspace which, in some sense, takes the place of $U \cup V$. Let then, by definition,

$$U + V \;=\; \left\{ u + v \in W \;\middle|\; u \in U,\, v \in V \right\} .$$

(a) Prove that $U + V$ is a subspace.
(b) Prove that, in fact, $U + V$ is the smallest subspace of W such that $U \cup V \subset U + V$.
(c) Suppose W is finite dimensional. Prove that

$$\dim(U + V) = \dim U + \dim V - \dim(U \cap V) \ .$$

Problem 2: As a follow-up from the previous problem, let W be a vector space, and let U, V be two subspaces of W. We say that W is the *direct sum* of U and V if $W = U + V$ and $U \cap V = \{0\}$. In this case we write $W = U \oplus V$. (Note: the concept of direct sum is a similar in many respect to the concept of linear independence).
(a) Prove that, if $W = U \oplus V$, then any element of W can be written in a *unique* way as a sum of an element of U and an element of V. (Actually, this is a necessary and sufficient condition).
(b) Suppose W is a finite dimensional vector space and $U \subset W$ is a subspace. Prove that there exists a subspace $U' \subset W$ such that $W = U \oplus U'$.

Problem 3: Consider the vector space $\mathbb{F}[x]$ of all polynomials in x with coefficients in $\mathbb{F}$. Given a polynomial $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$, we define its *degree* to be the largest power of x which appears. (For example, the degree of $p(x)$ equals n if $\alpha_n \neq 0$; more in general, the degree of $p(x)$ is the largest integer i such that $\alpha_i \neq 0$.)
(a) Consider the subset $A_3 \subset \mathbb{F}[x]$ of all polynomials of degree equal to 3. Is A_3 a subspace? (Justify your answer).
(b) Consider the subset $P_3 \subset \mathbb{F}[x]$ of all polynomials of degree less than or equal to 3. Is P_3 a subspace? (Justify your answer)

Problem 4: As a follow up from the previous problem,
(a) What is the dimension of P_3? (Justify your answer)
(b) Prove or disprove the following statement: there exists a basis of P_3 such that non of the basis element has degree 2.

Problem 5: Find all solutions (if any) of the following systems of linear equations.

(a)

$$\begin{aligned} x_1 + x_2 + x_3 &= 8 \\ x_1 + x_2 + x_4 &= 1 \\ x_1 + x_3 + x_4 &= 14 \\ x_2 + x_3 + x_4 &= 14 \end{aligned}$$

(b)

$$\begin{aligned} -x_1 + x_2 + x_4 &= 0 \\ x_2 + x_3 &= 1 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 1 \\ 3x_1 + x_3 &= 1 \\ 4x_1 + x_2 + 2x_3 &= 1 \end{aligned}$$