

## PROBLEM SET 5 - PROBLEM 5

### PART A

First, take an arbitrary  $P(x) \in \mathbf{R}[x]$ . Let  $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ .

$$\begin{aligned} D \circ I(P(x)) &= D \circ \int_0^x (a_0 + a_1y + a_2y^2 + \cdots + a_ny^n)dy \\ &= D \circ [(a_0x + a_1x^2 + a_2x^3 + \cdots + a_{n-1}\frac{1}{n}x^n + a_n\frac{1}{n+1}x^{n+1})] \\ &= \frac{d}{dx}(a_0x + a_1x^2 + a_2x^3 + \cdots + a_{n-1}\frac{1}{n}x^n + a_n\frac{1}{n+1}x^{n+1}) \\ &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \\ &= P(x) \end{aligned}$$

Therefore,  $D \circ I = \mathbf{I}$ .

$$\begin{aligned} I \circ D(P(x)) &= I \circ (0 + a_1 + 2a_2x \cdots na_nx^{n-1}) \\ &= \int_0^x (a_1 + 2a_2y \cdots na_ny^{n-1})dy \\ &= c + a_1x + a_2x^2 + \cdots + a_nx^n \\ &= P(x) + (c - a_0) \end{aligned}$$

Since the constant of integration  $c$  can be anything,  $I \circ D \neq \mathbf{I}$ .

### PART B

1. To show that  $V_0$  is a subspace of  $V$ , we first need to show that it is non-empty. Since  $P(0) = 0$ , we're done. Now, we need to show that it is closed on vector addition.

Let  $u, v \in V_0$ .  $P(u + v) = P(u) + P(v) = 0 + 0 = 0$ . Therefore,  $u + v \in V_0$ .

Now we need to show that it is closed under scalar multiplication. Let  $v \in V_0$  and  $\lambda \in F$ . Since  $P(v) = 0$ ,  $\lambda P(v) = 0$ . Therefore,  $P(\lambda v) = 0$ . So,  $V_0$  is a subspace.

To show that  $V_1$  is a subspace of  $V$ , we first need to check that it's non empty. We know that this is true since  $P(0) = 0$ .

To show that it is closed under vector addition, let  $u, v \in V_1$ . Therefore,  $P(u + v) = P(u) + P(v) = u + v$ .

To show closure under scalar multiplication, let  $v \in V_1$  and  $\lambda \in F$ .  $P(\lambda v) = \lambda P(v) = \lambda v$ . So,  $V_1$  is a subspace.

**2.** To show that  $V = V_0 \oplus V_1$ , we need to show 2 things:  $V = V_0 + V_1$  and  $V_0 \cap V_1 = \{0\}$ . Every  $v \in V$  can be written as  $v = (v - P(v)) + P(v)$ .  $P(v - P(v)) = P(v) - P(P(v)) = 0$ . Therefore  $(v - P(v)) \in V_0$ . And, since  $P(P(v)) = P(v)$ ,  $P(v) \in V_1$ . To show that  $V_0 \cap V_1 = \{0\}$ , if  $v \in V_0 \cap V_1$ , then  $P(v) = v$  and  $P(v) = 0$ . Therefore,  $v = 0$ .