MATH 23A

P-SET 5

Problem 1

A is an $m \times n$ matrix.

The goal of this problem is to show that $\dim \mathrm{Col}(A) = \dim \mathrm{Row}(A)$

(a) assume $t = \dim \mathrm{Row}(A)$

Since $\{r_1, \ldots, r_m\}$ is a spanning set of Row(A) by definition, it can be reduced to a basis with t elements. Reordering of rows will not affect Row(A), so let's now assume that $\{r_1, \ldots, r_t\}$ is a basis of Row(A)

(b) Consider $A' = \begin{bmatrix} r_1 \\ \vdots \\ r_t \end{bmatrix}$ (remember that A is just $\begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$)

Any solution of hom(A) will certainly be a solution of hom(A'), since any solution of hom(A) satisfies the equations given by ~~[illegible]~~ r_1 through r_t. ~~[illegible]~~

But in fact the equations given by r_{t+1} through r_m do not further restrict the solution set of hom(A), since these equations are represented by linearly dependent rows & thus simply represent $0 = 0$.

So ~~[illegible]~~ $\dim Sol(hom(A)) = \dim Sol(hom(A'))$.

(c) We know from class that if we have a set of homogeneous equations in n unknowns, then the dimension of the solutions is $n - r$, where r is the rank of the coefficient matrix.

So
$$\dim Sol(hom(A)) = n - rk(A)$$
$$\dim Sol(hom(A)) = n - rk(A') \implies rk(A) = rk(A')$$

) Examining A', we note that the columns lie in $\mathbb{F}^t$, so there can be at most t linearly independent columns. But $t = \dim \text{Row}(A)$

And $\dim \text{Col}(A') = \text{rk}(A') = \text{rk}(A) = \dim \text{Col}(A)$

$$\Rightarrow \quad \dim \text{Col}(A) \leq \dim \text{Row}(A)$$

) Consider A^T the transpose of A

Since we deduced parts (a) through (d) for an arbitrary matrix, the same process will tell us that

$$\dim \text{Col}(A^T) \leq \dim \text{Row}(A^T)$$

But A^T has the rows of A for columns and the columns of A for rows

$$\Rightarrow \quad Col(A^T) = Row(A) \quad \& \quad Row(A^T) = Col(A)$$

Substituting we have

$$\dim Row(A) \leq \dim Col(A)$$

But (d) $\Rightarrow$ $\dim Col(A) \leq \dim Row(A)$

$$\Rightarrow \dim Row(A) = \dim Col(A)$$

$\blacksquare$