Problem Set 5, Problem 3 Solution

(a) $\sim_I$ is not reflexive, since switching two rows of a matrix changes it.

$\sim_I$ is symmetric, since if $A \sim B$, one pair of rows were switched to change A to B, so the same pair can be switched back to change B to A.

$\sim_I$ is not transitive, since it is not possible to undo two row switches with only one.

$\sim_{II}$ is reflexive, since we can take any row to 1 times itself.

$\sim_{II}$ is symmetric, since $r_j \to \lambda r_j$ is inverted by $r_j \to (\lambda)^{-1} r_j$

$\sim_{II}$ is not transitive, since multiplying two different rows by constants cannot be undone in one step.

$\sim_{III}$ is reflexive, since we can take $r_i \to r_i + 0r_j$.

$\sim_{III}$ is symmetric, since $r_i \to r_i + \lambda r_j$ is inverted by $r_i \to r_i - \lambda r_j$.

$\sim_{III}$ is not transitive, since adding one row to two different other rows cannot be undone in one row operation.

(b) $\sim$ is reflexive since A is taken to A by 0 row operations.

If $A \sim B$, then we have by definition a series of row operations taking $A \to A_1 \to ... \to A_n \to B$. Since each row operation is symmetric, we can reverse this chain to get $B \to A_n \to ... \to A_1 \to A$, so $A \sim B$ implies $B \sim A$, and $\sim$ is symmetric.

If $A \sim B$ and $B \sim C$, then we have series $A \to A_1 \to ... \to B$ and $B \to B_1 \to ... \to C$, so we have series $A \to A_1 \to ... \to B \to B_1 \to ... \to C$, so $A \sim C$, and so $\sim$ is transitive.

Therefore $\sim$ is an equivalence relation.

(c) First, let A be the $1 \times m$ matrix

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

If all $a_j = 0$, then A is already in eschelon form. Otherwise, choose a non-zero a_j, switch it with the first element using ERO I, divide the new first row by a_j to get a 1 at the top, then subtract $a_k r_1$ from the other rows to get eschelon form

$$A = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

.

Therefore there is a matrix in eschelon form in the equivalence class of all $1 \times m$ matrices A.

Assume that A is $k \times m$ and can be converted into escelon form by EROs. Then, let A' be A with an additional column added.

$$A = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ a_{21} & \dots & a_{2k} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mk} \end{pmatrix}$$

$$A' = \begin{pmatrix} a_{11} & \dots & a_{1k} & a_{1,k+1} \\ a_{21} & \dots & a_{2k} & a_{2,k+1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mk} & a_{m,k+1} \end{pmatrix}$$

Change A to B through EROs (this is allowed by our assumptions; perform the same EROs on A' to get the matrix B' which is in eschelon form except for it's right hand column.

$$A' = \begin{pmatrix} B \mid c \end{pmatrix}$$

Where B is a matrix in eschelon form and

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Since B is in eschelon form, it has some number of rows with a 1 at the beginning, followed possibly by some rows that are all zeros. If no rows are all zeros, then B' is already in eschelon form. If rows r_i through r_m are all zero in B, then if the elements c_i through c_m are all zero B' is in eschelon form; if some c_j with $j \geq i$ is non-zero, switch rows j and i using ERO I, use ERO II to change the c_j to 1, then use ERO III to turn the elements c_{i+1} through c_m to 0. Then, we will have changed A' to eschelon form with EROs, so if it is possible for a $k \times m$ matrix it is possible for a $(k+1) \times m$ matrix as well. So, by induction, we can turn any $n \times m$ matrix into eschelon form through EROs.

(d) If $(x_1, x_2, \ldots, x_n)$ is a solution to the equations r_1 through r_m, it is a solution to the equations in different order (so the solution set is unaffected by ERO I), it is a solution to any sum of different equations (so the solution set is unaffected by ERO III), and it is a solution to a non-zero constant times any equation (so the solution set is unaffected by ERO II).

So, the EROs do not affect the solution set, so if $A \sim B$, then $sol(A * hom) = sol(B * hom)$. So, since $dim(Sol(A * hom)) = n - rk(A)$ and $dim(Sol(B*hom)) = n - rk(B)$, we can conclude that $n - rk(A) = n - rk(B)$, so $rk(A) = rk(B)$.

(e) We can make the following sequence:

$$A = \begin{pmatrix} 2 & -1 & 1 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 3 & -1 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 1 & 2 \\ 0 & \frac{3}{2} & -\frac{1}{2} & -2 \\ 0 & 3 & -1 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 3 & -1 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & -\frac{1}{6} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

= B

$Col(B)$ is clearly both generated and spanned by the basis $(1, 0, 0)$, $(0, 1, 0)$ (though many other bases are also fine), so $rk(B) = 2$