

Math 23a Theoretical Linear Algebra and Multivariable Calculus I

PROBLEM SET 5

Problem 1: Let A be an $m \times n$ matrix with real coefficients, i.e.

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdot & \cdots & \cdots & \cdot \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix}$$

Recall that the *columns* of A are, by definition, the m -column vectors,

$$c_1 = \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \cdot \\ \alpha_{m1} \end{bmatrix}, \quad c_2 = \begin{bmatrix} \alpha_{12} \\ \alpha_{22} \\ \cdot \\ \alpha_{m2} \end{bmatrix}, \quad \cdots, \quad c_n = \begin{bmatrix} \alpha_{1n} \\ \alpha_{2n} \\ \cdot \\ \alpha_{mn} \end{bmatrix} \in \mathbb{R}^m$$

Similarly, the *rows* of A are, by definition, the n -row vectors,

$$\begin{aligned} r_1 &= [\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n}] \\ r_2 &= [\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n}] \\ & \quad \cdots & \cdots & \cdots & \cdots \\ r_m &= [\alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn}] \end{aligned}$$

They are elements of the vector space \mathbb{R}_n of all n -row vectors (It is a space analogue to \mathbb{R}^n ; the only difference is that we take rows instead of columns. In particular $\dim \mathbb{R}_n = n$). We also define the *column space* of A to be the subspace of \mathbb{R}^m generated by the column vectors, i.e.

$$Col(A) = \text{span}\{c_1, c_2, \dots, c_n\} \subset \mathbb{R}^m$$

Similarly, the *row space* of A to be the subspace of \mathbb{R}_n generated by the row vectors, i.e.

$$Row(A) = \text{span}\{r_1, r_2, \dots, r_m\} \subset \mathbb{R}_n$$

In class we defined the *rank* of the matrix A to be the dimension of $Col(A)$. We want to show that this is the same as the dimension of $Row(A)$. In other words you need to prove the following

Theorem 1. $\dim Col(A) = \dim Row(A)$.

Hints:

- (a) If $t = \dim Row(A)$, prove that you can reorder the rows of the matrix A so that r_1, r_2, \dots, r_t is a basis of $Row(A)$.
- (b) Consider the new $t \times n$ matrix A' consisting of only the first t rows of the matrix A . Prove that the homogeneous systems of linear equations corresponding to the matrices A and A' have the exact same solutions.
- (c) Deduce that $rk(A) = rk(A')$.
- (d) Deduce that $\dim Col(A) \leq \dim Row(A)$.
- (e) By exchanging the roles of rows and columns (namely considering the *transpose matrix*), conclude that $\dim Col(A) = \dim Row(A)$.

Problem 2: Consider the 4×6 real matrix

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 0 & 3 & 2 & -4 \\ -1 & 0 & 1 & -1 & 3 & 6 \\ -1 & 1 & 1 & 2 & 3 & 1 \\ 3 & 1 & 0 & 3 & 6 & \alpha \end{bmatrix}$$

where α is some fixed real number. Let also A be the 4×5 matrix obtained by erasing the last column of \tilde{A} .

- Write down the columns $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}^4$ of the matrix A , and consider the space $Col(A) \subset \mathbb{R}^4$ defined in the previous problem.
- Find a sublist of $(c_1, c_2, c_3, c_4, c_5)$ which forms a basis of $Col(A)$. What is the rank of A ?
- Consider the homogeneous system of linear equations (**hom*) corresponding to the matrix A and let $Sol(*hom) \subset \mathbb{R}^5$ be the space of all solutions of (**hom*). Use Gauss elimination to find all the elements of $Sol(*hom)$.
- Find a basis of $Sol(*hom)$. What is its dimension?
- Check that the dimension formula, $\dim Sol(*hom) = n - rk(A)$, holds.
- Consider the (non-homogeneous) system of linear equation (***) corresponding to the augmented matrix \tilde{A} . Find all values of β for which (***) admits at least one solution.
- Use Gauss elimination to find all solutions of (***).
- Check that, give a solution $(\lambda_1, \dots, \lambda_5)$ of (***), every other solution is of the form $(\lambda_1 + \mu_1, \dots, \lambda_5 + \mu_5)$, where (μ_1, \dots, μ_5) is a solution of (**hom*).

Problem 3: Let A be an $m \times n$ matrix with real coefficients. In class we defined *elementary row operations* of the matrix A :

I: of type *I*, which consists in exchanging two rows of A ,

II: of type *II*, which consists in replacing the row r_i of the matrix A by $r_i + \lambda r_j$ for some $\lambda \in \mathbb{R}$ and some $j \neq i$.

III: of type *III*, which consists in replacing the row r_i of the matrix A by λr_i for some non-zero $\lambda \in \mathbb{R}$.

- Consider now the set $Mat_{mn}(\mathbb{R})$ of all $m \times n$. We define on this set the following relations $\sim_I, \sim_{II}, \sim_{III}$:
 - $A \sim_I B$ if and only if B is obtained by A by an elementary row operation of type *I*,
 - $A \sim_{II} B$ if and only if B is obtained by A by an elementary row operation of type *II*,
 - $A \sim_{III} B$ if and only if B is obtained by A by an elementary row operation of type *III*.

For each of the relations \sim_I, \sim_{II} and \sim_{III} say whether it is reflexive, symmetric and/or transitive (Justify your answer).

- We now define a new relation \sim on the set $Mat_{mn}(\mathbb{R})$ as follows: $A \sim B$ if and only if B is obtained starting with A and applying a finite number of elementary row operations. Prove that \sim is an equivalence relation.
- Prove, by induction on n , that in every equivalence class $[A]_{\sim}$ there is some matrix B in *normalized echelon* form, i.e. for every i the first

non-zero entry (if any) of row r_i of B is 1, and it is on the right of the first non-zero entry of the previous row r_{i-1} .

- (d) Prove that, if $A \sim B$, then $rk(A) = rk(B)$. (**Hint:** Consider the corresponding homogeneous systems of linear equations)
- (e) Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 3 & -1 & -4 \end{bmatrix}$$

Find a matrix $B \in [A]_{\sim}$ in normalized echelon form. Find a basis of $Col(B)$. Use this to compute $rk(A)$.

Problem 4: Recall from the exam that, given a vector space V and a subspace $U \subset V$, we define the equivalence relation \sim on V by

$$a \sim b \text{ if and only if } a - b \in U$$

Consider the corresponding partition, $V/U = \{[v] \mid v \in V\}$.

- (a) We define, on the set V/U the following operations of *scalar multiplication*

$$\lambda \cdot_{V/U} [v] = [\lambda \cdot_V v],$$

and of *addition*

$$[v] +_{V/U} [w] = [v +_V w].$$

Prove that these operations are well defined. i.e. their definition does not depend on the choice of the "representative" v of $[v]$.

- (b) Prove that V/U , with the above operations $\cdot_{V/U}$ and $+_{V/U}$, is a vector space.
- (c) Prove that, if V is finite dimensional, then $\dim V/U = \dim V - \dim U$.
- (d) Consider, as an example, the vector space (recall from Problem Set 4), $P_n \subset \mathbb{F}[x]$ of all polynomials in x with coefficients in \mathbb{F} of degree less than or equal to n . For $n \geq 3$, find a basis of the vector space P_n/P_2 . What is its dimension?

Problem 5a: Consider the vector space $\mathbb{R}[x]$ of all polynomials in x with real coefficients. We define the following linear transformations from $\mathbb{R}[x]$ to itself:

- $I : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ given by $I(P(x)) = \int_0^x dy P(y)$. (For example, if $P(x) = 1 + x - 2x^2$, then $I(P(x)) = x + \frac{1}{2}x^2 - \frac{2}{3}x^3$)
- $D : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ given by $D(P(x)) = \frac{d}{dx}P(x)$. (For example, if $P(x) = 1 + x - 2x^2$, then $D(P(x)) = 1 - 4x$)

Prove that $D \circ I = \mathbb{I}$ but $I \circ D \neq \mathbb{I}$.

Problem 5b: Recall from Problem Set 4 the definition of the direct sum $A \oplus B$ of subspaces A, B of a vector space V . Assume the linear map $P : V \rightarrow V$ is a *projection*, namely it is such that $P \circ P = P$. Define the sets:

$$V_0 = \{v \in V \mid P(v) = 0\}, \quad V_1 = \{v \in V \mid P(v) = v\}$$

- Prove that V_0 and V_1 are subspaces of V .
- Prove that $V = V_0 \oplus V_1$.

Extra Credit Problem: Prove in all details that the set $L(V, W)$ of all linear transformations from V to W is a vector space (with the operations $\cdot_{L(V,W)}$ and $+_{L(V,W)}$ defined in class)