Problem set 5, Problem #2
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Solution

a) $\operatorname{col}(A) = \operatorname{span}\left(\begin{bmatrix}1\\-1\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\3\\6\end{bmatrix}\right)$

b) You can check that c_1, c_2, c_3 are linearly independent.
Since

$$c_5 = \begin{pmatrix}2\\3\\3\\6\end{pmatrix} = 2\begin{pmatrix}1\\-1\\-1\\3\end{pmatrix} + 5\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}3\\-1\\2\\3\end{pmatrix} = -1\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + 3\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = c_4$$

$\Rightarrow$ c_1, c_2 + c_3 are a basis for col A, so
dim col $A = 3$.

c) $A = \begin{pmatrix}1 & 1 & 0 & 3 & 2\\ -1 & 0 & 1 & -1 & 3\\ -1 & 0 & 1 & 2 & 3\\ 3 & 1 & 0 & 3 & 6\end{pmatrix}$ = (after nasty row operations) $\begin{pmatrix}1 & 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 3 & 0\\ 0 & 0 & -1 & 1 & -5\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} = A'$ [you had to show these].

so $A\begin{pmatrix}x_1\\ \vdots\\ x_5\end{pmatrix} = 0$ implies $A'\begin{pmatrix}x_1\\ \vdots\\ x_5\end{pmatrix} = 0$, so

$$\begin{cases} x_1 + 2x_5 = 0 \\ x_2 + 3x_4 = 0 \\ -x_3 + x_4 - 5x_5 = 0 \end{cases}, \text{ which implies } \begin{cases} x_5 = -x_1/2 \\ x_4 = -x_2/3 \\ x_3 = 5x_1/2 - x_2/3 \end{cases}$$

Hence, sol (* hom) is all vectors of the form

★ $\{(\alpha, \beta, 5\alpha/2 - \beta/3, -\beta/3, -\alpha/2) \mid \alpha, \beta \in \mathbb{R}\}$

this has dimension 2 since it has 2 parameters

d) A basis is given by

$$\{(1, 0, 5/2, 0, 0, -1/2), (0, 1, 0, -1/3, -1/3, 0)\}.$$

You had to check that those are a basis.
(Get them by splitting the things in ★ in their α + β parts and factoring the parameters out)
$\Rightarrow$ dim (* sol hom) = 2

e) $\operatorname{rk} A = 3$, $n = 5$, dim sol (* hom) = 2.
Since $2 = 5 - 3$, dim sol (* hom) = $n - \operatorname{rk} A$.

g) We apply all operations done in c) to $\tilde{A}$ and get

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 & 0 & -5 \\ 0 & 0 & -1 & 1 & -5 & -7 \\ 0 & 0 & 0 & 0 & 0 & \alpha+2 \end{pmatrix}$$

(you had to either show them or say what I just said)

But further elimination gives

$$A'' = \begin{pmatrix} -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha+2 \end{pmatrix}$$

[you had to show it].

So if $A'' \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} = 0 \Rightarrow x_6 = 0$

But now sol* = sol* hom :)

h) you can either quote the theorem in class that essentialy does this <u>or</u> actually find the solutions.

COMMENT. [I didn't substract points for quoting the thm, but be advised that doing that is not really verifying.