Problem Set 6, Problem 3 Solution

Let V and W be finite dimensional vector spaces. Prove that V is W are isomorphic if and only if they have the same dimensions.

a) Assume $\dim V = \dim W$. Then, fix bases $\{v_1, v_2, \ldots, v_n\}$ for V and $\{w_1, w_2, \ldots, w_n\}$ for W. Define the linear mapping $T : V \to W$ by $T(a_1v_1 + a_2v_2 + \ldots a_nv_n) = a_1w_1 + a_2w_2 + \ldots a_nw_n$.

T is surjective since any $b_1w_1 + b_2w_2 + \ldots + b_nw_n \in W$ has pre-image $b_1v_1 + b_2v_2 + \ldots b_nv_n \in V$. T is surjective because $T(a_1v_1 + a_2v_2 + \ldots a_nv_n) \Rightarrow a_1w_1 + a_2w_2 + \ldots a_nw_n = 0 \Rightarrow a_1 = a_2 = \ldots = a_n = 0$ by the linear indepence of $w_1, w_2, \ldots, w_n$.

Therefore, T is a bijection between V and W, so they are isomorphic.

Alternate solution:
We showed in class that $V \cong F^n$, where $n = \dim V$. Then $V \cong F^n \cong W$. Since $\cong$ is an equivalence relation, this implies that $V \cong W$.

b) Assume V and W are isomorphic. Then, there exists a bijective linear map $T : V \to W$.

Then, since T is surjective, $Im(T) = W$, so $\dim(Im(T)) = \dim(W)$. Since T is injective, $\dim(\ker(T)) = 0$. By the rank-nullity theorem, $\dim(V) = \dim(\ker(T)) + \dim(Im(T)) = \dim(W)$.

Note:
It is a common misconception that the rank-nullity theorem implies that every vector in V is either in $\ker(T)$ or $Im(T)$. This is not true ($Im(T)$ is not even *in* V).