PROBLEM SET 6, PROBLEM 5

(a) $B=(e_1,\dots,e_n)$, $C=(v_1,\dots v_n)$

Let $P = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ be the change of basis matrix from B to C.

$$\Rightarrow \quad [e_1 \cdots e_n]\, Q = [v_1 \cdots v_n]$$

So, we have

$$v_1 = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + \cdots + a_{n1}e_n$$
$$v_2 = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + \cdots + a_{n2}e_n$$
$$\vdots$$
$$v_n = a_{1n}e_1 + a_{2n}e_2 + \cdots + a_{nn}e_n$$

we need to show that Q is the matrix of the linear transformation $I_V : V \to V$ where C is the starting and B is the target basis.

This is the matrix of I_V.

$$m(I_V) = \left[I_V(v_1) \,\middle|\, I_V(v_2) \,\middle|\, I_V(v_3) \,\middle|\, \cdots \,\middle|\, I_V(v_n) \right]$$

The k^{th} column is the representation of $I_V(v_k) = v_k$ in terms of $(e_1,\dots,e_n)$

$$\Rightarrow \quad I_V(v_k) = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix} \text{ w/respect to } e_1,\dots,e_n.$$

$$m(I_V) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} = Q$$

$\boxtimes$

(b)

$$\begin{tikzcd}
(e_1,\dots,e_n) \arrow[r] & V \arrow[r, "A"] \arrow[d, "Q"'] & W \arrow[d, "P"] & (f_1,\dots,f_m) \arrow[l] & V \arrow[r, "T"] & W \arrow[d, "I_W"] \\
(v_1,\dots,v_n) \arrow[r] & V \arrow[r, "B"] & W & (w_1,\dots,w_m) \arrow[l] & V \arrow[r, "T"] \arrow[u, "I_V"] & W
\end{tikzcd}$$

bases

We are given $\underset{④}{T} = \underset{③}{I_W} \circ \underset{②}{T} \circ \underset{①}{I_V}$

	starting basis	target basis
①	$(v_1,\dots,v_n)$	$(e_1,\dots,e_n)$
②	$(e_1,\dots,e_n)$	$(f_1,\dots,f_m)$
③	$(f_1,\dots,f_m)$	$(w_1,\dots,w_m)$
④	$(v_1,\dots,v_n)$	$(w_1,\dots,w_m)$

we have $T = I_W \circ T \circ I_V$

$$m(T) = m(I_W)\, m(T) \cdot m(I_V)$$

$$B = P \cdot A \cdot Q^{-1}$$

Another, more rigorous way is as follows:

$$T(v_1,\dots,v_n) = (w_1,\dots,w_m)\cdot B$$

$$\begin{aligned}
T(e_1,\dots,e_n) &= (I_W \cdot T \cdot I_V)(e_1,\dots e_n) \\
&= (I_W \cdot T)(I_V(e_1,\dots e_n)) \\
&= (I_W \cdot T)(v_1,\dots v_n)\cdot Q
\end{aligned}$$

$$= I_W(T(v_1, \dots v_n)) \cdot Q$$

$$= I_W(w_1, \dots w_m) \cdot B \cdot Q$$

$$= (f_1, \dots f_m) \cdot A$$

Because $T(e_1, \dots e_n) = (f_1, \dots f_m) \cdot A$

$$A = P^{-1} \cdot B \cdot Q$$

$$\Rightarrow B = PAQ^{-1}$$