Math 23aTheoretical Linear Algebra and Multivariable Calculus I

PROBLEM SET 6

Problem 1: Recall we denote by P_n the vector space of all polynomials in x with real coefficients of degree less than or equal to n. Consider the following linear functions:

$$I \ : \ P_2 \longrightarrow P_3 \ , \quad I(P(x)) \ = \ \int_1^x P(y)dy \ ,$$

and

$$D \ : \ P_3 \longrightarrow P_2 \ , \quad D(P(x)) \ = \ \frac{dP(x)}{dx} \ .$$

(a) Find a formula for the composite linear transformations: $I \circ D$ and $D \circ I$. (In particular, from which space to which space are they defined?)
(b) Fix the basis $\mathcal{B}_2 = (1, x, x^2)$ of P_2 and the basis $\mathcal{B}_3 = (1, x, x^2, x^3)$ of P_3. Compute the matrices A_I and A_D of I and D respectively (with respect to this choice of bases).
(c) Compute the matrices $A_{I\circ D}$ and $A_{D\circ I}$ of $I \circ D$ and $D \circ I$ respectively (by using the definition of $I \circ D$ and $D \circ I$).
(d) Check that $A_{I\circ D} = A_I \cdot A_D$ and $A_{D\circ I} = A_D \circ A_I$ (i.e. matrix multiplication corresponds to composition of linear transformations).

Problem 2: As a follow up of the Problem 1, consider now the new bases: $\mathcal{C}_2 = (1, x-1, (x-1)^2)$ of P_2 and $\mathcal{C}_3 = (1, x-1, (x-1)^2, (x-1)^3)$ of P_3.
(a) Find the matrix B_I of I with respect to these new bases.
(b) Find the matrix P of change of basis from $\mathcal{B}_2$ to $\mathcal{C}_2$, and the matrix Q of change of basis from $\mathcal{B}_3$ to $\mathcal{C}_3$.
(c) Verify that A_I and B_I are related by the appropriate change of basis formula.

Problem 3: Let V and W be finite dimensional vector spaces. Prove that V and W are isomorphic if and only if they have the same dimensions.

Problem 4: The purpose of this problem if to make you think on the meaning of the sentence: "isomorphic vector spaces can be thought of as the same vector space". Suppose V and V' are isomorphic vector spaces of dimension n, and let T be an isomorphism from V to V'.
(a) Prove that $(v_1, \dots, v_n)$ is a basis of V if and only if $(T(v_1), \dots, T(v_n))$ is a basis of V'.
(b) Consider a linear transformation $S : \ V \to U$. Define the "corresponding" linear transformation (and explain in which sense it "corresponds") $S' : \ V' \to U$, and prove that

$$T(\mathrm{Ker}(S)) = \mathrm{Ker}(S') \quad \text{and} \quad \mathrm{Im}(S) = \mathrm{Im}(S') \ .$$

(c) Consider a linear transformation $S : \ U \to V$. Define the "corresponding" linear transformation (and explain in which sense it "corresponds") $S' : \ U \to V'$, and prove that

$$\mathrm{Ker}(S) = \mathrm{Ker}(S') \quad \text{and} \quad T(\mathrm{Im}(S)) = \mathrm{Im}(S') \ .$$

Problem 5: Let V be a vector space of dimension n, and let $\mathcal{B} = (e_1, \ldots, e_n)$ and $\mathcal{C} = (v_1, \ldots, v_n)$ be two bases of V. We denote by I_V the identity linear transformation on V, defined by $I_V(v) = v$ for all $v \in V$.

(a) Prove that the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is the same as the matrix of the linear transformation $I_V : V \to V$ if we fix the basis $\mathcal{C}$ on the "starting" vector space V and the basis $\mathcal{B}$ on the "target" vector space V.

(b) Let $T : V \to W$ be a linear transformation from V to W. Use part (a) and the obvious identity $T = I_W \circ T \circ I_V$ to give a new proof of the change of basis formula $B = PAQ^{-1}$, discussed in class.