PSET 7, Problem 1

$\boxed{(a) \Rightarrow (b)}$ The columns of A are linearly independent, so we have n linearly independent vectors in $\mathbb{R}^n \Rightarrow$ the columns of A form a basis for $\mathbb{R}^n$.

$\boxed{(b) \Rightarrow (c)}$ The columns of A form a basis $\{c_1, \dots, c_n\}$ for $\mathbb{R}^n$.

Remember that in this basis, $c_1 = 1 \cdot c_1 + 0 \cdot c_2 + \dots + 0 \cdot c_n$, so it is represented by $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Thus, considering the matrix mult. $\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix} \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$,

We see that A maps the basis vector c_n to it's coordinates in the standard basis $\Rightarrow A$ is a change of basis from $\{c_1, \dots, c_n\}$ to $\{e_1, \dots, e_n\}$

$\boxed{(c) \Rightarrow (d)}$ Any change of basis matrix from $\{a_1, \dots, a_n\}$ to $\{b_1, \dots, b_n\}$ is invertible, since ~~the~~ the change of basis matrix from $\{b_1, \dots, b_n\}$ to $\{a_1, \dots, a_n\}$ is the inverse.

$\boxed{(d) \Rightarrow (e)}$ A invertible $\Rightarrow \exists A^{-1}$ such that

$$AA^{-1} = \mathbb{1}, \quad \text{so} \quad \det AA^{-1} = \det \mathbb{1} = 1$$

$$\det AA^{-1} = \det A \det A^{-1}$$

$$\Rightarrow \det A \neq 0$$

$\boxed{(e) \Rightarrow (f)}$ Remember that if A has linearly dependent rows, then $\det A = 0$.

By contrapositive, $\det A \neq 0 \Rightarrow$ the rows of A are linearly independent.

$\boxed{(f) \Rightarrow (g)}$ The rows of A are linearly independent, so we have n linearly independent vectors in $\mathbb{R}^n$

$\Rightarrow$ the rows form a basis.

$\boxed{(g) \Rightarrow (a)}$ Remember that row rank $=$ col. rank, so if the rows form a basis for $\mathbb{R}^n$, then

$n =$ row rank $=$ col. rank.

Thus the n columns of A span a space of dimension $n \Rightarrow$ they are linearly independent.

Since we can get to any statement from any other statement by this cycle, we have shown that (a), (b), (c), (d), (e), (f), and (g) are all equivalent ∎