Problem Set 7, Problem 3 Solution

a) If $T(a + bx + cx^2 + dx^3) = 0$, then $(a + d) + (b - d)x + (c + d)x^2 + (a + b + c + d)x^3 = 0$, so $a = -d, b = d$, and $c = -d$. Therefore, $\ker(T) = span\{1 - x + x^2 - x^3\}$, and $\dim \ker(T) = 1$.

$Im(T) = span\{1 + x^3, x + x^3, x^2 + x^3, 1 - x + x^2 + x^3\}$ by definition. Since $\dim \ker(T) = 1$ and $\dim P_3 = 4$, by the rank-nullity theorem, $4 = 1 + \dim Im(T) \Rightarrow \dim Im(T) = 3$.

$\dim P_3/\ker(T) = \dim P_3 - \dim \ker(T) = 3$ by a result from problem set 5.

b) We have the spanning set $\{1 + x^3, x + x^3, x^2 + x^3, 1 - x + x^2 + x^3\}$ for $Im(T)$ from (a). We know that $1 - x + x^2 + x^3$ is linearly dependent on the other 3, also from (a). Since $\dim(Im(T)) = 3$, the remaining three vectors must be a basis of $Im(T)$, so our basis is: $\{1 + x^3, x + x^3, x^2 + x^3\}$.

To make a basis for $P_3/\ker(T)$, we start with our basis for $\ker T$, $\{1 - x + x^2 - x^3\}$ and expand it to a basis for P_3. If we make the basis $\{1 - x + x^2 - x^3, v_1, v_2, v_3\}$ in this manner, the set $\{[v_1], [v_2], [v_3]\}$ is a basis for $P_3/\ker(T)$.

Proof that $\{[v_1], [v_2], [v_3]\}$ is a basis for $P_3/\ker(T)$:

Linear independence:
$a_1[v_1] + a_2[v_2] + a_3[v_3] = 0 \Rightarrow a_1v_1 + a_2v_2 + a_3v_3 = a(1 - x + x^2 - x^3)$ for some $a \in \mathbb{F}$. However, this is impossible unless $a = a_1 = a_2 = a_3 = 0$, since $\{1 - x + x^2 - x^3, v_1, v_2, v_3\}$ is a linearly independent set.

Since there are three independent vectors and $\dim(P_3/\ker(T)) = 3$, they must be a basis.

There were many ways to choose this basis, but the most popular acceptable answers were:
$\{[1], [x], [x^2]\}, \{[x], [x^2], [x^3]\}, \{[1 + x^3], [x + x^3], [x^2 + x^3]\}$.

c) To calculate the matrix A, we just calculate $\bar{T}([v_j])$ for each of our basis vectors, and see what it is in terms of the basis vectors of $Im(T)$. Different

answers were arrived at depending on the choice of basis.

$\{[1 + x^3], [x + x^3], [x^2 + x^3]\}$

$\bar{T}([1 + x^3]) = 2 - x + x^2 + 2x^3 = \mathbf{2}(1 + x^3) + \mathbf{(-1)}(x + x^2) + \mathbf{1}(x^2 + x^3)$
$\bar{T}([x + x^3]) = 1 + x^2 + 2x^3 = \mathbf{1}(1 + x^3) + \mathbf{0}(x + x^3) + \mathbf{1}(x^2 + x^3)$
$\bar{T}([x^2 + x^3]) = 1 - x + 2x^2 + 2x^3 = \mathbf{1}(1 + x^3) + \mathbf{(-1)}(x + x^3) + \mathbf{2}(x^2 + x^3)$

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix}$$

$\{[1], [x], [x^2]\}$: (calculations excluded for these two cases)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\{[x], [x^2], [x^3]\}$:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

d) $\det(A)$ came out to different numbers depending on choice of basis; it was 2, 1, and 1 in the three cases shown above. We know *a priori* that it will not be 0, however, because by problem 2 $\bar{T}$ is an isomorphism.