

Math 23a Theoretical Linear Algebra and Multivariable Calculus I

PROBLEM SET 7

Problem 1: Let A be an $n \times n$ matrix. Prove that the following statements are equivalent:

- (a) The columns of A are linearly independent.
- (b) The columns of A form a basis of \mathbb{R}^n .
- (c) A is a change of basis matrix (from which basis to which basis of \mathbb{R}^n ?).
- (d) A is invertible (i.e. there exists an $n \times n$ matrix A^{-1} such that $AA^{-1} = A^{-1}A = \mathbb{I}$).
- (e) $\det A \neq 0$.
- (f) The rows of A are linearly independent.
- (g) The rows of A form a basis of \mathbb{R}^n .

Problem 2: Recall from Problem Set 5 that, given a subspace U of a vector space V , we defined the "quotient vector space" V/U .

Given a linear transformation $T : V \rightarrow W$, let $K = \text{Ker } T \subset V$ and $I = \text{Im } T \subset W$, and consider the quotient vector space V/K . Prove that there is a well defined linear transformation $\bar{T} : V/K \rightarrow I$ given by $\bar{T}([v]) = Tv$. (Note: it's not obvious that this function is well defined!) Moreover, prove that this map establishes an isomorphism of vector spaces $V/K \simeq I$.

Problem 3: As an example of what showed in Problem 2, consider the linear transformation

$$T : P_3 \longrightarrow P_3$$

given by (recall P_3 denotes the space of polynomials in x of degree less than or equal than three):

$$\begin{aligned} T(a + bx + cx^2 + dx^3) \\ = a(1 + x^3) + b(x + x^3) + c(x^2 + x^3) + d(1 - x + x^2 + x^3) . \end{aligned}$$

- (a) Find $\text{Ker } T$ and $\text{Im } T$. What is the dimension of $\text{Ker } T$, $\text{Im } T$ and $P_3/\text{Ker } T$?
- (b) Fix a basis of $P_3/\text{Ker } T$ and a basis of $\text{Im } T$.
- (c) Consider the linear transformation $\bar{T} : P_3/\text{Ker } T \rightarrow \text{Im } T$ defined in Problem 2. Write the matrix A of \bar{T} in the bases you chose in part (b).
- (d) Compute $\det A$ and check that $\det A \neq 0$ (why do we know a priori that $\det A$ is not zero?)

Problem 4: Consider an $(m + n) \times (m + n)$ matrix M of this form

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is an $m \times m$ matrix, B is an $m \times n$ matrix, C is an $n \times n$ matrix and 0 denote the $n \times m$ matrix with all zero entries. Prove that

$$\det M = \det A \det C .$$

(**Hint:** use elementary row operations on the first m rows to reduce A in upper triangular form and then compute the determinant using column expansion).

More in general, prove by induction that if a matrix M has a "block diagonal form"

$$M = \begin{bmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{bmatrix}$$

where $A_1, A_2 \dots A_n$ are square matrices, then $\det M = \det A_1 \det A_2 \dots \det A_n$.

Problem 5: Prove or disprove the following statements (disprove means find a counterexample):

- (a) $\det(A + B) = \det A + \det B$,
- (b) $\det((A + B)^2) = (\det(A + B))^2$,
- (c) $\det((A + B)^2) = \det(A^2 + 2AB + B^2)$,
- (d) $\det(A + B) = \det(B + A)$,
- (e) $\det(AB) = \det(BA)$,
- (f) if A is an invertible matrix, then $\det(A^{-1}) = (\det A)^{-1}$.