PROBLEM 2, PROBLEM SET 7. GERARDO CON DIAZ

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This question is really close to some of my favorite things in math, so I'm quite excited about it :) First we will show that $\bar{T}$ is well-defined. To do this, we need to show that, for any w equivalent to v under the equivalence relation we are given, $\bar{T}[v] = \bar{T}[w]$.
However, if w v then $w - v \in KerT$, so there is some $k \in KerT$ such that $w = v + k$. It follows that $\bar{T}[w] = \bar{T}[v+k] = T(v+k) = T(v) + T(k) = T(v) = \bar{T}[v]$. We just showed that the map is well-defined.
The map $\bar{T}$ establishes an isomorphism between V/K and I by sending a given $[v] \in V/K$ to $T(v)$. First note that for any $v \in V$, the element $[v] \in V/K$ is mapped to $T(v)$ by $\bar{T}$, so the map is certainly surjective.
Also note that if $\bar{T}[v] = \bar{T}[w]$, then $T(v) = T(w)$, so that $T(v - w) = 0$. But then $v - w \in KerT$, so v w. This means that $[v] = [w]$, so the map is injective.
Finally, to show that it is linear, it suffices to see that vor vectors v and w and any scalar a, $\bar{T}[v + aw] = T(v + aw) = T(v) + T(aw) = T(v) + aT(w) = \bar{T}[v] + a\bar{T}[w]$, so the map is linear.
This shows that we have an isomorphism $V/K \cong I$.