

### Problem Set 8, Problem 3 Solution

a) In the space of  $n \times n$  matrices, let the relation  $A \sim B$  denote that  $A$  is similar to  $B$ .

$A = IAI^{-1}$  since  $I = I^{-1}$ . Therefore  $A \sim A$ , and  $\sim$  is reflexive.

Assume  $A \sim B$ , so  $B = PAP^{-1}$  for some matrix  $P$ . Then,  $A = P^{-1}BP = QBQ^{-1}$ , if we let  $Q = P^{-1}$ . Therefore,  $A \sim B \Rightarrow B \sim A$ , so  $\sim$  is symmetric.

Assume  $A \sim B$  and  $B \sim C$ , so  $B = PAP^{-1}$  and  $C = QBQ^{-1}$ . Then  $C = Q(PAP^{-1})Q^{-1} = (PQ)A(Q^{-1}P^{-1})$ . Since  $(PQ)(Q^{-1}P^{-1}) = I$ ,  $(PQ)^{-1} = Q^{-1}P^{-1}$ , and therefore  $A \sim C$ . Therefore  $\sim$  is transitive.

Since  $\sim$  is reflexive, transitive, and symmetric, it is an equivalence relation in the space of  $n \times n$  matrices.

As we will show in (b), matrices are equivalent if and only if their associated linear transformations are, so we can conclude that  $\sim$  is an equivalence relation in the space of linear transformations as well.

(b) Assume that  $S$  and  $T$  are similar linear transformations, so  $T = LSL^{-1}$  for some linear transformation  $L$ . With respect to a basis  $\{v_1, \dots, v_n\}$  for  $V$ , let the matrices for  $L, S,$  and  $T$  be  $P, A,$  and  $B$ , respectively.

As shown in class, for any basis,  $Mat(T_1 \circ T_2) = Mat(T_1)Mat(T_2)$  for any linear transformations  $T_1$  and  $T_2$ . Also,  $(Mat(T))^{-1} = Mat(T^{-1})$ . Therefore, the relation  $T = LSL^{-1}$  implies that  $Mat(T) = Mat(L)Mat(S)Mat(L^{-1})$ , so  $B = PAP^{-1}$  and  $B \sim A$ .

Likewise, if  $A$  and  $B$  are similar matrices, so  $B = PAP^{-1}$ , if we let  $L, S,$  and  $T$  be the linear transformations represented by  $P, A,$  and  $B$  according to the basis  $\{v_1, \dots, v_n\}$ , using the above reasoning we see that  $B = PAP^{-1} \Rightarrow T = LSL^{-1}$ , so  $S \sim T$ .

Therefore,  $A \sim B \Leftrightarrow S \sim T$ .

(c) Let  $S \sim T$ , so  $T = LSL^{-1}$ . Assume  $\lambda$  is an eigenvalue of  $S$  with eigenvector  $v$ .

Then,  $Sv = \lambda v$ . Therefore,  $T(Lv) = (LSL^{-1})(Lv) = LSv = \lambda Lv$ . So,  $Lv$  is an eigenvector of  $T$  with eigenvalue  $\lambda$ .

Therefore,  $S$  and  $T$  have the same eigenvalues; furthermore, for any given eigenvector  $v$  of  $S$ ,  $Lv$  is an eigenvector of  $T$  with the same eigenvalue.