PROBLEM SET 8, PROBLEM 5

Let $\mathbb{K}$ denote the set of all 2×2 matrices of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} ;\ a, b \in \mathbb{R}.$$

The function $f: \mathbb{C} \longrightarrow \mathbb{K}$ is given precisely by

$$f(a+bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

This function f is surjective because a, b can take up any value in $\mathbb{R}$ if $a+bi \in \mathbb{C}$. It is invertible because its inverse

$$f^{-1}\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) = a+bi \text{ is well-defined.}$$

There fore, the function is bijective.

Showing that $f(z \underset{\mathbb{C}}{+} z') = f(z) \underset{\mathbb{K}}{+} f(z')$

Let $z = a+bi$, $z' = c+di$

$$f(z \underset{\mathbb{C}}{+} z') = f((a+bi)+(c+di)) = f((a+c)+(b+d)i) = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix}$$

$$= \begin{bmatrix} a+c & -b-d \\ b+d & a+c \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

$$= f(z) \underset{\mathbb{K}}{+} f(z')$$

Showing that $f(z \underset{\mathbb{C}}{\cdot} z') = f(z) \underset{\mathbb{K}}{\cdot} f(z')$

$$f(z \underset{\mathbb{C}}{\cdot} z') = f\big((a+bi)\cdot(c+di)\big) = f\big((ac-bd)+(ad+bc)i\big)$$

$$= \begin{bmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

$$= f(z) \underset{\mathbb{K}}{\cdot} f(z')$$

Note: You did not need to prove that $\mathbb{K}$ is a field.

The existence of a bijective function $f: \mathbb{C} \longrightarrow \mathbb{K}$ which preserves the field structure automatically means that $\mathbb{K}$ must be a field.