

Math 23a Theoretical Linear Algebra and Multivariable Calculus I

PROBLEM SET 8

Problem 1: Find all eigenvalues and eigenvectors for each of the following matrices (considered as linear transformations $\mathbb{R}^2 \rightarrow \mathbb{R}^2$):

$$(a) \begin{bmatrix} 4 & 5 \\ -1 & -2 \end{bmatrix}, \quad (b) \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}, \quad (c) \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}.$$

Problem 2: Suppose that λ is an eigenvalue for the linear transformation $T : V \rightarrow V$, where V is a finite dimensional vector space (over \mathbb{R} or \mathbb{C}).

- (a) Show that λ^2 is an eigenvalue for $T \circ T$.
- (b) If T is invertible, show that λ^{-1} is an eigenvalue for T^{-1} . (What happens if $\lambda = 0$?)

Problem 3: Let V be a finite dimensional vector space. Two linear transformations S, T from V to V are said to be *similar* if there exists an invertible linear transformation $L : V \rightarrow V$ such that $T = LSL^{-1}$. In the same way, two $n \times n$ matrices A, B are said to be *similar* if there exists an invertible $n \times n$ matrix P such that $B = PAP^{-1}$.

- (a) Prove that "being similar" is an equivalence relation (both in the space of operators of V and in the space of $n \times n$ matrices).
- (b) In a given basis (v_1, \dots, v_n) of V , let A be the matrix of S and let B be the matrix of T . Prove that S and T are similar linear transformations if and only if A and B are similar matrices.
- (c) Prove that, if S and T are similar linear transformations, then they have the same eigenvalues. What is the relation among eigenvectors?

Problem 4: Let $V = \mathbb{R}^3$, and consider the linear map $L : V \rightarrow V$ given by

$$L \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y + z \\ 2y + 3z \\ 4z \end{bmatrix}.$$

- (a) Write the matrix A for L in the standard basis of \mathbb{R}^3 .
- (b) Find the eigenvalues of L and find a basis of V consisting of eigenvectors of L .
- (c) Write the matrix B for L in the basis of eigenvectors.
- (d) Find the change of basis matrix P from the standard basis to the basis of eigenvectors, and show directly that A and B are related by the change of basis formula (Note: in particular, A and B are similar.)

Problem 5: We say that two fields \mathbb{K} and \mathbb{F} are *isomorphic* if there exists a bijective function $f : \mathbb{K} \rightarrow \mathbb{F}$ which "preserves the field structure", i.e. such that $f(a +_{\mathbb{K}} b) = f(a) +_{\mathbb{F}} f(b)$, $f(a \cdot_{\mathbb{K}} b) = f(a) \cdot_{\mathbb{F}} f(b)$ for all $a, b \in \mathbb{K}$. (In this case we think at \mathbb{K} and \mathbb{F} as being the "same" field. Note, in particular, that $f(0_{\mathbb{K}}) = 0_{\mathbb{F}}$ and $f(1_{\mathbb{K}}) = 1_{\mathbb{F}}$.)

In this problem we want to give a new description of the field of complex numbers. Namely we want to prove the following

Theorem 1. *The field \mathbb{C} is isomorphic to the set of all 2×2 matrices*

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad a, b \in \mathbb{R},$$

with the usual operations of addition and multiplication of matrices.