Math 23aTheoretical Linear Algebra and Multivariable Calculus I

PROBLEM SET 9

Problem 1: A) Recall we defined in class the complex conjugate of a complex number $z = \alpha + \beta i$ to be $\bar{z} = \alpha - \beta i$. In this (and the next) problem, consider $\mathbb{R}$ as a subset of $\mathbb{C}$ by thinking at a real number $x \in \mathbb{R}$ as the complex number $x + 0i \in \mathbb{C}$. Similarly, we consider $\mathbb{R}^n$ as a subset of $\mathbb{C}^n$ (but NOT a subspace! Why?) Prove that:

(a) $z\bar{z} \in \mathbb{R}$, $z + \bar{z} \in \mathbb{R}$, $\frac{1}{i}(z - \bar{z}) \in \mathbb{R}$,

(b) $\overline{zw} = \bar{z}\bar{w}$, $\overline{z + w} = \bar{z} + \bar{w}$.

B) Similarly, given $v = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$, we define $\bar{v} = \begin{bmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{bmatrix} \in \mathbb{C}^n$, called the complex conjugate vector, and given an $n \times n$ matrix with complex entries $A = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ & \cdots & \\ z_{n1} & \cdots & z_{nn} \end{bmatrix}$, we define $\bar{A} = \begin{bmatrix} \bar{z}_{11} & \cdots & \bar{z}_{1n} \\ & \cdots & \\ \bar{z}_{n1} & \cdots & \bar{z}_{nn} \end{bmatrix}$. Prove that

(a) $v + \bar{v} \in \mathbb{R}^n$, $\frac{1}{i}(v - \bar{v}) \in \mathbb{R}^n$,

(b) $\overline{Av} = \bar{A}\bar{v}$, $\overline{v + w} = \bar{v} + \bar{w}$.

Problem 2: Let A be an $n \times n$ matrix with real entries. We can think at the same matrix A in two ways: as a linear transformation $A_{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}^n$, and as a linear tranformation $A_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^n$. Recall that $A_{\mathbb{C}}$ has at least one eigenvalue $\lambda \in \mathbb{C}$, and let $v \in \mathbb{C}^n$ be an eigenvector of $A_{\mathbb{C}}$ with eigenvalue λ.

(a) Prove that $\bar{\lambda}$ is also an eigenvalue of $A_{\mathbb{C}}$ (with which eigenvector?)

(b) If $\lambda = \alpha + 0i$, $\alpha \in \mathbb{R}$, prove that there exists an eigenvector $u \in \mathbb{R}^n$ of $A_{\mathbb{R}}$ with eigenvalue α.

(c) If $v = \begin{bmatrix} x_1 + y_1 i \\ \vdots \\ x_n + y_n i \end{bmatrix} \in \mathbb{C}^n$, let $v_1 = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $v_2 = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$.
Prove that $U = \mathrm{span}(v_1, v_2) \subset \mathbb{R}^n$ is an invariant subspace with respect to $A_{\mathbb{R}}$.

Problem 3: Let $T : V \to V$ be an operator on the vector space V over $\mathbb{R}$ of dimension $n \geq 1$. Prove the following

Theorem 1. *There exists a non-zero subspace $U \subset V$, invariant with respect to the action of T, such that* $\dim U \leq 2$.

(**Hint:** fix any basis $(v_1, \ldots, v_n)$ of V, and consider the matrix A of T; use the result of Problem 2.)

Problem 4: Let A be an $n \times n$ matrix with complex entries of the following form

$$A = \begin{bmatrix} \lambda & & * \\ & \ddots & \\ 0 & & \lambda \end{bmatrix} .$$

(a) Prove that, for every $N \geq 1$, the matrix A^N has the form

$$A^N = \begin{bmatrix} \lambda^N & & * \\ & \ddots & \\ 0 & & \lambda^N \end{bmatrix} .$$

(b) Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{C}^n$. Prove that, for every $i = 1, \dots, n$, we have $e_1, e_2, \dots, e_i \in \operatorname{Ker}((A - \lambda \mathbb{1})^i)$, and that $\operatorname{Ker}((A - \lambda \mathbb{1})^n) = \mathbb{C}^n$.

(c) Prove that, if $\lambda = 0$, then $A^N = 0$ for every $N \geq n$.

Problem 5: Let $s \geq 1$, let $\lambda_1, \dots, \lambda_s$ be distinct complex numbers, let $n_1, \dots, n_s$ be positive integers, and let $n = n_1 + \cdots + n_s$. Let A be an $n \times n$ matrix with complex entries of the following form

$$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{bmatrix} ,$$

i.e. A is "block diagonal", where for each $i = 1, \dots, s$, A_i is an $n_i \times n_i$ matrix of the following form

$$A_i = \begin{bmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix} .$$

Consider, as usual, A as a linear transformation $A : \mathbb{C}^n \to \mathbb{C}^n$. If $(e_1, \dots, e_n)$ is the standard basis of $\mathbb{C}^n$, consider the following subspaces of $\mathbb{C}^n$:

$$\begin{aligned} U_1 &= \operatorname{span}(e_1, \dots, e_{n_1}) , \\ U_2 &= \operatorname{span}(e_{n_1+1}, \dots, e_{n_1+n_2}) , \\ &\dots \\ U_s &= \operatorname{span}(e_{n_1+\cdots+n_{s-1}+1}, \dots, e_n) . \end{aligned}$$

Prove that:

(a) $\mathbb{C}^n = U_1 \oplus \cdots \oplus U_s$

(b) For every $N \geq 1$,

$$A^N = \begin{bmatrix} A_1^N & & 0 \\ & \ddots & \\ 0 & & A_s^N \end{bmatrix} ,$$

(c) $\operatorname{Ker}(A - \lambda_i \mathbb{1})^N = U_i$ for every $N \geq n_i$.

Extra Credit (+10pts) Deduce that, if $T : V \to V$ is an operator on a complex vector space V and $\lambda \in \mathbb{C}$ is an eigenvalue of T, then the algebraic multiplicity $n(\lambda)$ of λ (i.e. the multiplicity of λ as root of the characteristic polynomial)

$$n(\lambda) = \dim \operatorname{Ker}((T - \lambda \mathbb{1})^{\dim V}) \ .$$

Moreover, if $\lambda_1, \ldots, \lambda_s$ are all the eigenvalues of T, then

$$V = \operatorname{Ker}((T - \lambda_1 \mathbb{1})^{\dim V}) \oplus \cdots \oplus \operatorname{Ker}((T - \lambda_s \mathbb{1})^{\dim V}) \ .$$