Problem Set 6, Problem 4

a) ($\Rightarrow$) Assume $(v_1, \dots, v_n)$ is a basis for V.

- $Tv_1, \dots, Tv_n$ are linearly independent:

  $\alpha_1 Tv_1 + \dots + \alpha_n Tv_n = 0 \Rightarrow T\alpha_1 v_1 + \dots + T\alpha_n v_n = 0 \Rightarrow T(\alpha_1 v_1 + \dots + \alpha_n v_n) = 0.$

  But T is an isomorphism, so its kernel is trivial. Therefore, $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$ and, since $(v_1, \dots, v_n)$ is a basis, $\alpha_1 = \dots = \alpha_n = 0$.

- Since $V \simeq V'$, $\dim V = \dim V' = n$. Because $(Tv_1, \dots, Tv_n)$ are n linearly independent vectors in V', they form a basis.

($\Leftarrow$) Assume $(Tv_1, \dots, Tv_n)$ is a basis for V'.

- $v_1, \dots, v_n$ are linearly independent:

  $\alpha_1 v_1 + \dots + \alpha_n v_n = 0_V \Rightarrow T(\alpha_1 v_1 + \dots + \alpha_n v_n) = 0_{V'} \Rightarrow T\alpha_1 v_1 + \dots + T\alpha_n v_n = 0 \Rightarrow$

  $\alpha_1 Tv_1 + \dots + \alpha_n Tv_n = 0 \Rightarrow \alpha_1 = \dots = \alpha_n$ since $Tv_1, \dots, Tv_n$ is a basis.

- $\dim V = \dim V' = n$, so any set of n linearly independent vectors is a basis. We're done.

b) $T: V \to V'$, an isomorphism $\Rightarrow \exists\, T^{-1}: V' \to V$, an isomorphism.

$S: V \to U$, linear.

Define $S': V' \to U$, $S'(v') = S \circ T^{-1}(v')$.

It is useful to draw:

$$\begin{tikzcd} V' \arrow[r, "T^{-1}"] \arrow[rr, bend right, "S'"'] & V \arrow[r, "S"] & U \end{tikzcd}$$

The two transformations (S and S') are "corresponding" in the sense that, as we will show, $\operatorname{Im} S = \operatorname{Im} S'$ and, furthermore, to each v in the domain of S corresponds a vector v' in the domain of S' such that $Sv = S'v'$. This is just another way to describe the bijection between V and V'.

- $T(\ker(S)) = \ker(S')$

  if $v' \in \ker S' \Rightarrow S \circ T^{-1}(v') = 0_U \Rightarrow T^{-1}(v') \in \ker(S)$

  $\Rightarrow T(T^{-1}(v')) \in T(\ker(S)) \Rightarrow v' \in \ker(S')$.

conversely, if $v' \in T(\ker(S)) \Rightarrow v' = Tv$ for some $v \in V$ with $Sv = 0_U$. By the definition of T^{-1}, $T^{-1}(v') = v$

$\Rightarrow 0 = Sv = S(T^{-1}(v')) = S'(v')$. So $v' \in \ker(S')$.

$\Rightarrow \ker(S') = T(\ker(S))$.

- $Im(S) = Im(S')$

  If $u \in Im(S) \Rightarrow \exists\, v \in V$ such that $S(v) = u$. Then $S'(T^{-1}v) = u$,
  so $u \in Im(S')$

  If $u' \in Im(S') \Rightarrow \exists\, v' \in V'$ such that $S'(v') = u'$. Then $S(Tv') = u$,
  so $u \in Im(S)$.

c) Again $T : V \to V'$ (an isomorphism) and $S : U \to V$.
This time we want $S' : U \to V'$.
Define $S'(u) = T \circ S(u)$.
Again it might be useful to draw:

$$\begin{tikzcd} U \arrow[r, "S"'] \arrow[rr, bend left, "S'"] & V \arrow[r, "T"'] & V' \end{tikzcd}$$

- $\ker(S) = \ker(S')$

  if $u \in \ker(S) \Rightarrow S'(u) = T(S(u)) = T(0) = 0 \Rightarrow u \in \ker(S')$.

  if $u \in \ker S' \Rightarrow T(S(u)) = 0$. But T is an isomorphism $\Rightarrow \ker(T) = \{0\}$

  $\Rightarrow S(u) = 0_V \Rightarrow u \in \ker(S)$.

- $T(Im(S)) = Im(S')$

  $v' \in Im(S') \Leftrightarrow \exists\, u \in U$ such that $S'(u) = v' \Leftrightarrow \exists\, u \in U$ such that $T(S(u)) = v' \Leftrightarrow v' \in T(Im(S))$.

We say that S and S' are "corresponding" because they have the same domain, U, and there is a bijection between the sets to which S and S' map U.