

Problem Set 6, Problem 4

a) (\Rightarrow) Assume (v_1, \dots, v_n) is a basis for V .

• Tv_1, \dots, Tv_n are linearly independent:

$$\alpha_1 Tv_1 + \dots + \alpha_n Tv_n = 0 \Rightarrow T\alpha_1 v_1 + \dots + T\alpha_n v_n = 0 \Rightarrow T(\alpha_1 v_1 + \dots + \alpha_n v_n) = 0.$$

But T is an isomorphism, so its kernel is trivial. Therefore,

$$\alpha_1 v_1 + \dots + \alpha_n v_n = 0 \text{ and, since } (v_1, \dots, v_n) \text{ is a basis, } \alpha_1 = \dots = \alpha_n = 0.$$

• Since $V \cong V'$, $\dim V = \dim V' = n$. Because (Tv_1, \dots, Tv_n) are n linearly independent vectors in V' , they form a basis.

(\Leftarrow) Assume (Tv_1, \dots, Tv_n) is a basis for V' .

• v_1, \dots, v_n are linearly independent:

$$\alpha_1 v_1 + \dots + \alpha_n v_n = 0_V \Rightarrow T(\alpha_1 v_1 + \dots + \alpha_n v_n) = 0_{V'} \Rightarrow T\alpha_1 v_1 + \dots + T\alpha_n v_n = 0 \Rightarrow$$

$$\alpha_1 Tv_1 + \dots + \alpha_n Tv_n = 0 \Rightarrow \alpha_1 = \dots = \alpha_n \text{ since } Tv_1, \dots, Tv_n \text{ is a basis.}$$

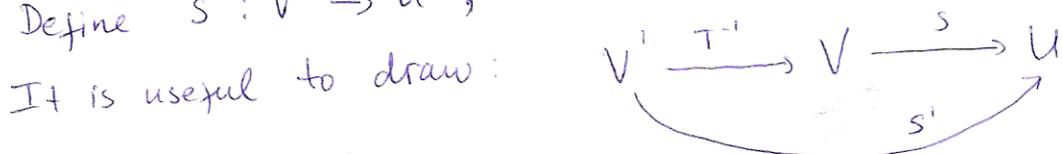
• $\dim V = \dim V' = n$, so any set of n linearly independent vectors is a basis. We're done.

b) $T: V \rightarrow V'$, an isomorphism $\Rightarrow \exists T^{-1}: V' \rightarrow V$, an isomorphism.

$S: V \rightarrow U$, linear.

Define $S': V' \rightarrow U$, $S'(v') = S \circ T^{-1}(v')$.

It is useful to draw:



The two transformations (S and S') are "corresponding" in the sense that, as we will show, $\text{Im } S = \text{Im } S'$ and, furthermore, to each v in the domain of S corresponds a v' vector v' in the domain of S' such that $Sv = S'v'$. This is just another way to describe the bijection between V and V' .

• $T(\ker(S)) = \ker(S')$

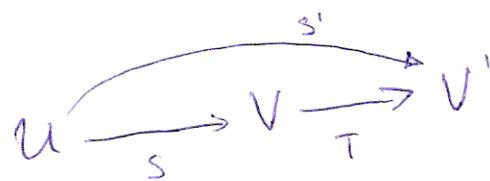
$$\text{if } v' \in \ker S' \Rightarrow S \circ T^{-1}(v') = 0_U \Rightarrow T^{-1}(v') \in \ker(S)$$

$$\Rightarrow T(T^{-1}(v')) \in T(\ker(S)) \Rightarrow v' \in \ker(S'). \quad \checkmark$$

conversely, if $v' \in T(\ker(S)) \Rightarrow v' = Tv$ for some $v \in V$ with $Sv = 0_u$. By the definition of T^{-1} , $T^{-1}(v') = v \Rightarrow 0 = Sv = S(T^{-1}(v')) = S'(v')$. So $v' \in \ker(S')$.
 $\Rightarrow \ker(S') = T(\ker(S))$.

• $\text{Im}(S) = \text{Im}(S')$
 If $u \in \text{Im}(S) \Rightarrow \exists v \in V$ such that $S(v) = u$. Then $S'(T^{-1}v) = u$, so $u \in \text{Im}(S')$.
 If $u' \in \text{Im}(S') \Rightarrow \exists v' \in V'$ such that $S'(v') = u'$. Then $S(Tv') = u'$, so $u' \in \text{Im}(S)$.

c) Again $T: V \rightarrow V'$ (an isomorphism) and $S: U \rightarrow V$.
 This time we want $S': U \rightarrow V'$.
 Define $S'(u) = T \circ S(u)$.
 Again it might be useful to draw:



• $\ker(S) = \ker(S')$
 if $u \in \ker(S) \Rightarrow S'(u) = T(S(u)) = T(0) = 0 \Rightarrow u \in \ker(S')$.
 if $u \in \ker(S') \Rightarrow T(S(u)) = 0$. But T is an isomorphism $\Rightarrow \ker(T) = \{0\}$
 $\Rightarrow S(u) = 0_v \Rightarrow u \in \ker(S)$.

• $T(\text{Im}(S)) = \text{Im}(S')$

$v' \in \text{Im}(S') \Leftrightarrow \exists u \in U$ such that $S'(u) = v' \Leftrightarrow \exists u \in U$ such

that $T(S(u)) = v' \Leftrightarrow v' \in T(\text{Im}(S))$.

We say that S and S' are "corresponding" because they have the same domain, U , and there is a bijection between the sets to which S and S' map U .