On Friday I did not have time to finish the computations of the integral of the following integral. Let

$\Sigma = \{x_1, x_2, x_3 | x_3 > 0, x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3, \mathcal{O}$ be the ccw orientation of Σ *when you are looking from inside = from below*,$\omega = x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2$. Find $\int_{\Sigma,\Omega} \omega$

a)First of all we have to choose a parametrization $\alpha : D \to \Sigma, D \subset \mathbb{R}^2$ compatible with Ω.

b)The second step is to compute $\eta := \alpha^*(\omega) \in \Omega^2(D)$. In other words we have to find a function $f(x,y), (x,y) \in D$ such that $\eta = f(x,y) dx \wedge dy$.

c)The third step is to compute the integral $\int_D f(x,y) dxdy$

A solution.

a)As we have seen we can choose $D = \{(x,y), x^2 + y^2 < 1\}, \alpha(x,y) = (y, x, \sqrt{1-(x^2+y^2)})$.

b)I'll present two ways to compute the function $f(x,y)$

1) By the definition we have $f(x,y) = \eta(x,y)(f_1, f_2)$ where (f_1, f_2) is the standard basis of $\mathbb{R}^2$. By the definition of $\alpha^*(\omega)$ we have $\eta(x,y)(f_1,f_2) = \omega(x,y)(D\alpha(x,y)(f_1), D\alpha(x,y)(f_2))$. Since $\alpha(x,y) = (y, x, \sqrt{1-(x^2+y^2)})$ we have (?)

$$D\alpha(x,y)(f_1) = e_2 - x/\sqrt{1-(x^2+y^2)} e_3$$

$$D\alpha(x,y)(f_2) = e_1 - y/\sqrt{1-(x^2+y^2)} e_3$$

Therefore

$$f(x,y) = \omega(\alpha(x,y))(e_2 - x/\sqrt{1-(x^2+y^2)} e_3, e_1 - y/\sqrt{1-(x^2+y^2)} e_3)$$

In other words

$$f(x,y) = ye^2 \wedge e^3 - xe^1 \wedge e^3 + \sqrt{1-(x^2+y^2)} e^1 \wedge e^2 ((e_2 - x/\sqrt{1-(x^2+y^2)} e_3, e_1 - y/\sqrt{1-(x^2+y^2}$$

$= I' + I'' + I'''$ where

$$I' = ye^2 \wedge e^3 (e_2 - x/\sqrt{1-(x^2+y^2)} e_3, e_1 - y/\sqrt{1-(x^2+y^2)} e_3) = -y^2/\sqrt{1-(x^2+y^2)},$$

$$I'' = -xe^1 \wedge e^3 (e_2 - x/\sqrt{1-(x^2+y^2)} e_3, e_1 - y/\sqrt{1-(x^2+y^2)} e_3) = -x^2/\sqrt{1-(x^2+y^2)}$$

and

$$I''' = \sqrt{1-(x^2+y^2)} e^1 \wedge e^2 ((e_2 - x/\sqrt{1-(x^2+y^2)} e_3, e_1 - y/\sqrt{1-(x^2+y^2)} e_3) = -\sqrt{1-(x^2+y}$$

So we have $f(x,y) = -(y^2/\sqrt{1-(x^2+y^2)} + x^2/\sqrt{1-(x^2+y^2)} + \sqrt{1-(x^2+y^2)} = -1/\sqrt{1-(x^2+y^2)}$.

2) Another computation of $f(x,y)$ [the one which was presented in class]

We have $\alpha^*x_1 = y, \alpha^*x_2 = x, \alpha^*x_3 = \sqrt{1-(x^2+y^2)}$. Therefore $\alpha^*dx_1 = d\alpha^*x_1 = dy, \alpha^*dx_2 = d\alpha^*x_2 = dx$ and $\alpha^*dx_3 = d\alpha^*x_3 = -(xdx+ydy)/\sqrt{1-(x^2+y^2)}$. So

$$\eta = -ydx\wedge(xdx+ydy)/\sqrt{1-(x^2+y^2)}+xdy\wedge(xdx+ydy)/\sqrt{1-(x^2+y^2)}+\sqrt{1-(x^2+y^2)}dx\wedge d$$

$$= -1/\sqrt{1-(x^2+y^2)}dx\wedge dy$$

So $f(x,y) = -1/\sqrt{1-(x^2+y^2)}$.

c)We see that

$$\int_{\Sigma,\Omega}\omega = \int_D -1/\sqrt{1-(x^2+y^2)}dxdy$$

If you introduce polar coordinates you find that

$$\int_D -1/\sqrt{1-(x^2+y^2)}dxdy = -2\pi\int_0^1 r/\sqrt{1-r^2}dr =$$

$$-\pi\int_0^1 1/\sqrt{1-t}dt = -2\pi$$

where $t = r^2$