Let V be a vector space, $\omega \in \Omega^1(V)$ be a differential 1-form. We define a function $f_\omega \in \Omega^0(V)$ by $f(v) := \int_{C(v),\mathcal{O}} \omega$ where $C(v), \mathcal{O}$ is the straight line from $\bar{0}$ to v in V oriented in the direction from $\bar{0}$ to v.

1. a)Show that any $v \in V$ we have $df_\omega(v)(v) = \omega(v)$

b) Show that in the case when $\omega$ is 1-form such that $d\omega = 0$ we have $\omega = df_\omega$

c) Let $U \subset V$ be an open set such that there exists a point $u_0 \in U$ such that for any $u \in U$ we have $C(u_0, u) \subset U$ where $C(u_0, u)$ is the interval from $u_0$ to u. Show that for any 1-form $\omega \in \Omega^1(U)$ such that $d\omega = 0$ there exists a smooth function $f_\omega \in \Omega^0(U)$ such that $\omega = df_\omega$

d)Let $U = \mathbb{R}^2 - (0,0), \omega := \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \Omega^1(U)$. Check that $d\omega = 0$ and prove that there is no function $f \in \Omega^0(U)$ such that $\omega = df$

2. Show that for any $\omega \in \Omega^i(\mathbb{R}^n)$ we have $d(d\omega) = 0$

3. Let $V$ be a vector space $U \subset V$ an open subset, $\omega' \in \Omega^{i'}, \omega'' \in \Omega^{i''}$ be differential forms on U.

Show that $d(\omega' \wedge \omega'') = d(\omega') \wedge \omega'' + (-1)^{i'} \omega' \wedge d\omega''$

A hint. First consider the case when $i', i'' \leq 1$

Let $V, V'$ be vector spaces $U \subset V, U' \subset V'$ open subsets and $\phi : U' \to U$ a smooth map. For any i we have defined the map $\phi^* : \Omega^i(U) \to \Omega^i(U')$ by

$$\phi^*(\omega)(u')(v'_1, ..., v'_i) := \omega(u)(v_1, ..., v_i), v_k := D\phi(u')(v'_k), u := \phi(u')$$

4. Show that for any $\omega \in \Omega^i(U)$ we have $d(\phi^*(\omega)) = \phi^*(d\omega)$

A hint. First consider the case when $i \leq 1$

5.Problems from the book

Chapter 15 Section 5 Numbers 3 and 6

Chapter 16 Section 4 Numbers 1 ,3,5

Don't forget to choose an orientation on manifolds before you compute integrals.