Definition. If V, W are vector spaces $U \subset V$ is an open subset, $f : U \to W$ a function. We say that f is *smooth* if it is infinitely differentiable [that is for any $n > 0$ f is n-times differentiable.

Theorem-Definition.

Let Σ be a subset of $\mathbb{R}^2, \sigma_0 \in \Sigma$. We say that Σ is *a smooth curve* at σ_0 if the following conditions $Q1, Q2, Q3$ are satisfied. Let $\sigma_0 = (x_0, y_0)$

$Q1$. Either there exists $\epsilon > 0$ and a smooth function $f : (x_0 - \epsilon, x_0 + \epsilon) \to \mathbb{R}$ such that $f(x_0) = y_0, (x, f(x)) \in \Sigma$ for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$ and there exists an open set $U \subset \mathbb{R}^2, \sigma_0 \in U$ such that for any $\sigma \in \Sigma \cap U$ there exists $x \in (x_0 - \epsilon, x_0 + \epsilon)$ such that $\sigma = (x, f(x))$ or

there exists $\epsilon > 0$ and a smooth function $g : (y_0 - \epsilon, y_0 + \epsilon) \to \mathbb{R}$ such that $g(y_0) = x_0, (g(y), y) \in \Sigma$ for all $y \in (y_0 - \epsilon, y_0 + \epsilon)$ and there exists an open set $U \subset \mathbb{R}^2, \sigma_0 \in U$ such that for any $\sigma \in \Sigma \cap U$ there exists $y \in (y_0 - \epsilon, y_0 + \epsilon)$ such that $\sigma = (g(y), y)$ or both.

If Σ satisfies $Q1$ at $\sigma \in \Sigma$ we define the tangent line $l\sigma \subset \mathbb{R}^2$ to Σ at σ as follows. If we can find $f : (x_0 - \epsilon, x_0 + \epsilon) \to \mathbb{R}$ such that $f(x_0) = y_0, (x, f(x)) \in \Sigma$ for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$ then we define the tangent line $l\sigma$ by the equation

$$l\sigma = \{(x, y) \in \mathbb{R}^2 | y - y_0 = f'(x_0)(x - x_0)$$

.

If a smooth function $g : (y_0 - \epsilon, y_0 + \epsilon) \to \mathbb{R}$ such that $g(y_0) = x_0, (g(y), y) \in \Sigma$ for all $y \in (y_0 - \epsilon, y_0 + \epsilon)$ then we define the tangent line $l\sigma$ by the equation

$$l\sigma = \{(x, y) \in \mathbb{R}^2 | (y - y_0)g'(y_0) = (x - x_0)$$

.

$Q2$. There exists an open set $U \subset \mathbb{R}^2, \sigma_0 \in U$ and a a smooth function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $F(\sigma_0) = 0, D_F(\sigma_0) \neq 0$ and $r_0 > 0$ such that for any $r, 0 < r < r_0$ we have $\Sigma \cap B_{\sigma_0}(r) = \{u \in B_{\sigma_0} | F(u) = 0\}$. [In other words $\Sigma \cap B_{\sigma_0}(r) = F^{-1}(0) \cap B_{\sigma_0}(r)$] where $B_{\sigma_0}(r) := \{u \in U | \|u - \sigma_0\| < r\}$.

If Σ satisfies $Q2$ at $\sigma \in \Sigma$ we define the tangent line $l\sigma \subset \mathbb{R}^2$ to Σ at σ as follows. Consider the affine function $F_1 : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F_1(x, y) := \partial F/\partial x(\sigma)(x - x_0) + \partial F/\partial y(\sigma)(y - y_0)$$

. Then we define the tangent line $l\sigma$ by the equation

$$l\sigma = \{(x, y) \in \mathbb{R}^2 F_1(x, y) = 0$$

$Q3$. There exists $\epsilon > 0$ and a smooth function $\phi : (-\epsilon, \epsilon) \to \mathbb{R}^2$ such that $\phi(0) = \sigma_0, D_\phi(0) \neq 0$ and $r_0 > 0$ such that for any $r, 0 < r < r_0$

such that we have $\Sigma \cap B_{\sigma_0}(r) = \phi(-\epsilon,\epsilon) \cap B_{\sigma_0}(r)$ where $\phi(-\epsilon,\epsilon) := \{\phi(x)\}, x \in (-\epsilon,\epsilon)$. Moreover for any $\delta, 0 < \delta < \epsilon$ we can find $r_0 > 0$ such that for any $r, 0 < r < r_0$ such that we have $\Sigma \cap B_{\sigma_0}(r) = \phi(-\delta,\delta) \cap B_{\sigma_0}(r)$.

If Σ satisfies $Q3$ at $\sigma \in \Sigma$ we define the tangent line $l\sigma \subset \mathbb{R}^2$ to Σ at σ as the image of $\mathbb{R}$ under an affine map $\phi_1 : \mathbb{R} \to \mathbb{R}^2, \phi_1(t) := \sigma + tD_\phi(0)$.

One can show that the tangent line $l\sigma$ is well defined [that is does not depend on a choice of a definition Qi or a choice of functions F [in the case of $Q2$] and ϕ [in the case of $Q3$]

One can ask whether it is necessary to demand that *for any* $\delta, 0 < \delta < \epsilon$ we can find $r_0 > 0$ such that for any $r, 0 < r < r_0$ such that we have $\Sigma \cap B_{\sigma_0}(r) = \phi(-\delta,\delta) \cap B_{\sigma_0}(r)$. Analyze the following example.

Consider the map $\phi = (\phi_1, \phi_2) : \mathbb{R} \to \mathbb{R}^2$ where

$$\phi_1(t) = \frac{-3t^2(t-1)}{3t^2 - 3t + 1}$$

$$\phi_2(t) = \frac{3t(t-1)^2}{3t^2 - 3t + 1}$$

Let $\Sigma := Im(\phi)$. $[Im(\phi) := \{\phi(t), t \in \mathbb{R}]$. Show that

a)$\Sigma = \{(x,y)|x^3 + y^3 - 3xy = 0\}$

b) Σ is smooth at all points $\sigma \in \Sigma, \sigma \neq (0,0)$.

c)Find whether Σ is smooth at $(0,0)$.

d)Check that the 2-d and 3-d definitions give the same answer only if we impose the condition that *for any* $\delta, 0 < \delta < \epsilon$ we can find $r_0 > 0$ such that for any $r, 0 < r < r_0$ such that we have $\Sigma \cap B_{\sigma_0}(r) = \phi(-\delta,\delta) \cap B_{\sigma_0}(r)$. (In other words, if we delete the sentence begining with “Moreover” in definition Q3, we get a different answer to whether Σ is continuous than if we use the whole of definition Q3.)