Homework for February 26-th

Definition. If V, W are vector spaces $U \subset V$ is an open subset, $f : U \to W$ a function. We say that f is *smooth* if it is infinitely differentiable [that is for any $N > 0$ f is N-times differentiable.

Theorem-Definition.

Let Σ be a subset of $\mathbb{R}^n, \sigma_0 \in \Sigma$. We say that Σ is *smooth hypersurface* at σ_0 if one of the following three conditions $Q1, Q2, Q3$ is satisfied.

Let $\sigma_0 = (x_0^i), 1 \leq i \leq n$. We define $\sigma_0^i \in \mathbb{R}^{n-1}$ by $\sigma_0^i := (x_0^1, ..., x_0^{i-1}, x_0^{i+1}, ..., x_0^n)$.

$Q1$.There exists $\epsilon > 0, i, 1 \leq i \leq n$ an open set $U \subset \mathbb{R}^{n-1}, \sigma_0^i \in U$ and a smooth function $f_i : U \to \mathbb{R}$ such that

A)$f(\sigma_0^i) = x_0^i$

B) for any $x = (x^1, ..., x^{n-1}) \in U$ we have $(x^1, ..., x^{i-1}, f(x), x^{i+1}, ..., x^n) \in \Sigma$ and

C)there exists an open set $U \subset \mathbb{R}^n, \sigma_0 \in U$ such that for any $\sigma \in \Sigma \cap U$ there exists $x = (x^1, ..., x^{n-1}) \in U$ such that $\sigma = (x^1, ..., x^{i-1}, f(x), x^{i+1}, ..., x^n)$.

$Q2$. There exists an open set $U \subset \mathbb{R}^n, \sigma_0 \in U$ and a smooth function $F : \mathbb{R}^n \to \mathbb{R}$ and a number $r_0 > 0$ such that

A)$F(\sigma_0) = 0$,

B)$D_F(\sigma_0) \neq 0$

C) for any $r, 0 < r < r_0$ we have $\Sigma \cap B_{\sigma_0}(r) = \{u \in B_{\sigma_0} | F(u) = 0\}$. [In other words $\Sigma \cap B_{\sigma_0}(r) = F^{-1}(0) \cap B_{\sigma_0}(r)$] where $B_{\sigma_0}(r) := \{u \in U | \|u - \sigma_0\| < r\}$.

$Q3$. There exists an open set $U \subset \mathbb{R}^{n-1}$, a smooth function $\phi : U \to \mathbb{R}^n$ and a number $r_0 > 0$ such that

A)$\phi(0) = \sigma_0$,

B)The linear map $D_\phi(0) : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is an imbedding

C) for any $r, 0 < r < r_0$ we have $\Sigma \cap B_{\sigma_0}(r) = \phi(U) \cap B_{\sigma_0}(r)$ where $\phi(U) := \{\phi(x)\}, x \in (U)$ and

D)for any open set $U' \subset U$ we can find $r_0' > 0$ such that for any $r, 0 < r < r_0'$ such that we have $\Sigma \cap B_{\sigma_0}(r) = \phi(U') \cap B_{\sigma_0}(r)$

1. a)Prove the conditions $Q1$ and $Q2$ are equivalent.

b)Show that $Q1 \Rightarrow Q3$.

c) Prove that $Q3 \Rightarrow Q1$ assuming the validity of the following result.

For any $i, 1 \leq i \leq n$ we denote by $p_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection $p(x_1, ..., x_n) := (x_1, ..., x_{i-1}, x_{i+1}, ...x_n)$.

Claim. Let $T : \mathbb{R}^{n-1} \to \mathbb{R}^n$ be an imbedding. Then there exists $i, 1 \leq i \leq n$ such that the composition $p_i \circ T \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is invertible.

d) Prove the Claim

2. a)Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $f'(t) \neq 0$ for all $t \in \mathbb{R}, V := Im(f) \subset \mathbb{R}$. Show that the map $f : \mathbb{R} \to V$ is a *homeomorphism*. That is show the existence of a continuous function $g : V \to \mathbb{R}$ which is inverse to f. [That is $f \circ g(v) = v$ for all $v \in V$ and $g \circ f(t) = t$ for all $t \in \mathbb{R}$.

3)Let $f_1(x, y) = e^x cos(y), f_2(x, y) = e^x sin(y)$ and $f = (f_1(x, y), f_2(x, y)) : \mathbb{R}^2 \to \mathbb{R}^2, V := Im(f)$.

a)Describe the set V

b)Show that for any $u = (x, y)$ the linear map $D_f(u)$ is invertible.

c) Show that the map $f : \mathbb{R}^2 \to V$ is not one-to-one.

4) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with continuous second derivatives. Let $u_0 \in \mathbb{R}^2$ be a point such that $D_f(u_0) = 0$ and $\partial^2 f/\partial^2 x(u_0) = \partial^2 f/\partial^2 y(u_0) = 1$ and $\partial^2 f/\partial x \partial y(u_0) = 0$. Show the existence of a number $\epsilon > 0$ such that for all $u = (x, y)|0 < x^2 + y^2 < \epsilon$ we have $f(u) > 0$