First the problems from the book Chapter 11

Section 1. Problems 5, 11, 12, 15

Section 2 . Problems 1 [please note that I don't assume that the function $g(x)$ is monotone], 10 ,11, 12

In addition do the following two problems

1. Let f, g be bounded functions on $[a, b]$. Show that

a)$\int_a^{b+}(f(x)+g(x))dx \leq \int_a^{b+} f(x) + \int_a^{b+} g(x)dx$

b)$\int_a^{b+} cf(x) = c\int_a^{b+} f(x)$ if $c > 0$ and

$\int_a^{b+} cf(x) = c\int_a^{b-} f(x)$ if $c < 0$.

Let $S \subset \mathbb{R}^2$ be the circle $S := (x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1$. Consider the map $\phi : \mathbb{R} \to S$ defined in the following way. For any $t \in \mathbb{R}$ let l_t be the line through the point $(-1, 0)$ with the equation $y = tx + t$. We define the point $\phi(t) \in \mathbb{R}^2$ as the second point of intersection of the line l_t with the curve S.

2. a) Show that ϕ defines a one-to-one map from $\mathbb{R}$ to $S - (-1, 0)$.

b) Construct the inverse map $\phi^{-1} : S - (-1, 0) \to \mathbb{R}$

c)Find all points $(x, y) \in S$ such that x and y are rational numbers.

d)Use the map ϕ to compute the integral $\int_a^b \frac{1}{\sqrt{1-x^2}}dx, 0 < a < b < 1$

e)Let $R(x, y)$ be any rational function, $Q(x) = x^2+\alpha x+\beta$ a quadratic polynomial. Show that the integral $\int_a^b R(x, \sqrt{Q(x)})dx$ is expressible in terms of elementary functions [such as $\lg, \arctan, \sec$ e.t.c

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, $a \in \mathbb{R}$ and $P_k^a(x)$ be taylor polynomial of degree k at a, $P_k^a(x) = \sum_{0 \leq i \leq k} c_i(x-a)^i$

a)Show that $f(x) - P_k^a(x) = 1/k! \int_0^h (h-t)^k f^{(k+1)}(a+t)dt$ where $h := x - a$

A hint. Use the integration by part.

b)Prove the existence of $c \in [a, x]$ such that $f(x) - P_k^a(x) = 1/(k+1)! f^{(k+1)}(c)h^{(k+1)}$