Definition 1 Let $\Lambda \subset \mathbb{R}^n$ be a set,

i)We say that Λ *is of content zero* if for any $a > 0$ we can cover Λ with a finite number of rectangles of the total area $< a$.

ii) Let a be a positive real number. We say that *the content of* $\Lambda < a$ [where a is a positive real number] if we can cover Λ with a finite number of rectangles of the total area $< a$.

iii). We say that *the measure of* Λ *is equal to zero* if for any $a > 0$ we can find rectangles $R_i, 1 \leq i$ (this is a possibly infinite collection of rectangles which cover C and such that $\sum_{i \geq 1} Area(R_i) < a$.

1 a)Let $\Lambda \subset \mathbb{R}^n$ be a compact set. Show that the measure of Λ is equal to zero iff for any $a > 0$ the content of Λ is smaller then a.

b) Show that a compact set of measure zero has content zero.

Definition 2 We say Let f be a bounded function on a rectangular subset $C \subset \mathbb{R}^n$. For any $\epsilon > 0$ we define C_ϵ as the set of all $c \in C$ such that there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in C$ such that $|x - c|, |y - c| < \delta$. We define $D_\epsilon := C - C_\epsilon$

2. a)Show that the set D_ϵ is closed.

Let $D(f) := \bigcup D_\epsilon \epsilon > 0$.

b) Show that a function f is continuous iff $D(f) = \emptyset$.

c)Show that a bounded function f is integrable iff for any $a > 0 \exists \epsilon > 0$ such that the content of $D_\epsilon < a$.

d)Prove [using problem 1] that a buonded function f is integrable iff the set $D(f)$ has measure zero.

e) Let $C = [0, 1]$ and $f : C \to \mathbb{R}$ be the function as in the problem 15 section 1 of the chapter 11. Describe the sets D_ϵ and D.

3) Let f, g be bounded integrable functions on a rectangle $C \subset \mathbb{R}^n$.

a)Show that the function $F := f + g$ on C is integrable and

$$\int_C F(x) dx_1....dx_n = \int_C f(x) dx_1....dx_n + \int_C g(x) dx_1....dx_n$$

.

b) Show that the function $G(x) := f(x)g(x)$ on C is integrable.

4) Let $C \subset \mathbb{R}^n$ be a compact smooth hypersurface. Show that C has content zero.

Integrals and limits

Definition 3. Let f_n a be sequence of functions on a subset C of $\mathbb{R}^n$. We say that the sequence f_n is *uniformly convergent* to a function f on C if for any $\epsilon > 0$ there exists n such that for all $m > n, c \in C$ we have $|f(c) - f_m(c)| < \epsilon$.

Last Friday we proved the following result.

Theorem. Let $C \subset \mathbb{R}^n$ be a compact,f_n a sequence of integrable functions which converges uniformly to a function f. Then the function f is integrable and the sequence $\int_C f_n(c)dx_1....dx_n$ converges to $\int_C f(c)dx_1....dx_n$.

4.Show that the conclusion of the Theorem could be false if the set C is not compact.

Improper integrals Until the last lecture we considered only integrals of the form $\int_C f(\bar{x})dx_1...dx_n$ where C is a bounded subset of $\mathbb{R}^n$ and f is a bounded function. But often we have to consider more general situation when neither the set C nor the function f are bounded. For simplicity of notations we assume that $C = \mathbb{R}^n$.

Assume first that the function f is non-negative. For any $R > 0$ we define $C_R := \{(x_1, ..., x_n)||x_i| \leq R, 1 \leq i \leq n$ and by f_R the function on C_R given by $f_R(c) := \min(f(c), R), c \in C_R$..

Definition 4. A non-negative function f on $\mathbb{R}^n$ is *improperly integrable* if all the functions f_R are integrable and

$$\sup_R \int_{C_R} f_R(x)dx_1....dx_n$$

exists. In this case we define

$$\int_{\mathbb{R}^n} f(x)dx_1....dx_n := \sup_R \int_{C_R} f_R(x)dx_1....dx_n$$

If $f : \mathbb{R}^n \to \mathbb{R}$ be any function we define functions $f^+, f^- : \mathbb{R}^n \to \mathbb{R}$ by $f^+(x) := \max(f(x), 0), f^-(x) := \max(-f(x), 0)$. It is clear that functions $f^\pm$ are non-negative. It is clear that $f = f^+ - f^-$.

Definition 5. A function f on $\mathbb{R}^n$ is *absolutely integrable* if the functions $f^\pm$ are integrable. In this case we define $\int f dx_1....dx_n := \int f^+ dx_1....dx_n - \int f^- dx_1....dx_n$

5. Let f, g be absolutely integrable [not necessarily bounded] functions on $[0, 1]$.

a)Show that the function $F := f + g$ on $[0, 1]$ is integrable and

$$\int_0^1 F(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx$$

.

b) Is it true that the function $G(x) := f(x)g(x)$ on C is necessarily integrable?

6. Let $C \subset \mathbb{R}^2$ be the square $C = \{(x, y)|0 \leq x, y \leq 1\}$. For any numbers $a, b > 0$ we define a function $f_{a,b} : C \to \mathbb{R}$ by $f_{a,b}(x, y) := \frac{1}{x^a y^b}$

if $xy \neq 0$, $f(x, y) = 0$ if $xy = 0$. Find for which pairs a, b the function $f_{a,b}$ is integrable.

Hint. Use the Fubini Theorem