Let V be a finite-dimensional vector space, $U \subset V$ and open set. We denote by $\Omega^0(U)$ the space of smooth functions $f : U \to \mathbb{R}$. We denote by $\Omega^1(U)$ the space of smooth functions $\omega : U \to V^\vee$. Given $f \in \Omega^0(U), \omega \in \Omega^1(U)$ we define $f\omega \in \Omega^1(U)$ by $f\omega(u) := f(u)\omega(u)$. Given $\omega', \omega'' \in \Omega^1(U)$ we define $\omega' + \omega'' \in \Omega^1(U)$ by $\omega' + \omega''(u) := \omega'(u) + \omega''(u)$.

For any $f \in \Omega^0(U)$ we define $df \in \Omega^1(U)$ by $df(u) \in V^\vee$ by $df(u) := Df(u) : V \to \mathbb{R}$.

1. Check that for any $f, g \in \Omega^0(U)$ we have $d(fg) = fdg + gdf$

Let $e_1, ..., e_n$ be a basis of V and $e^1, ..., e^n$ be the dual basis of $V^\vee$. We define functions $x^i, 1 \leq i \leq n$ on V by $x^i(v) := e^i(v)$

2. Show that the 1-form $dx^i : V \to V^\vee$ is given by $dx^i(v) = e^i$ for all $v \in V$.

Let V', V be vector spaces and $U' \subset V', U \subset V$ be open subsets and $\phi : U' \to U$ a smooth map. We define the map $\phi^{*0} : \Omega^0(U) \to \Omega^0(U')$ by $\phi^*(f) := f \circ \phi$.

3. Show that there exists unique linear map $\phi^{*1} : \Omega^1(U) \to \Omega^1(U')$ such that

i)$\phi^{*1}(f\omega) = \phi^{*0}(f)\phi^{*1}(\omega)$ and

b)$\phi^{*1}(df) = d\phi^{*0}(f)$ for all $f \in \Omega^0(U), \omega \in \Omega^1(U)$

A hint. You can define the map $\phi^{*1} : \Omega^1(U) \to \Omega^1(U')$ by

$$(*)\phi^{*1}(\omega)(u') := \omega(u) \circ D\phi(u') : V' \to \mathbb{R}$$

for $\omega \in \Omega^1(U)$. Then you have to check that

a)Such a map map is linear and satisfies the conditions i),ii) and

b)Any linear map $\Omega^1(U) \to \Omega^1(U')$ which satisfies the conditions i),ii coincides with the one given by $(*)$.

4. Compute the 1-forms $\phi^{*1}(f\omega)$ in the following cases

a)$U = V = \mathbb{R}^n, U' = V' = \mathbb{R}, \omega = \sum_1^n x^i dx^i$ and $\phi : V' \to V$ is given by $\phi(t) = (t, t^2, ..., t^n), t \in \mathbb{R}$

b)$U = V = \mathbb{R}^3, U' = V' = \mathbb{R}^2, \omega = xdy + ydz + zdx, \phi : V' \to V$ is given by $\phi(u, v) = (u^2, uv, v^2)$

Let $C \in V$ be a smooth curve and $\alpha : [a, b] \to C$ be a *parametrization* of C. That is we have a smooth map $\alpha : [a, b] \to C$ such that $d\alpha(t) \neq 0$ for all $t \in [a, b]$ and the map $\alpha : [a, b] \to C$ is one-to-one and onto. A parametrization of C defines an *orientation* of C. That is the direction in which we run through C.

4. a)Let $\alpha : [a, b] \to C, \alpha[a', b'] \to C$ be two parametrization of C. Show that α, α' define the same orientation on C iff $g'(t) > 0, t \in [a, b]$ where the function $g : [a, b] \to [a', b']$ is given by $g(t) := \alpha'^{-1}(\alpha(t))$. (You may assume for the sake of this problem and parts b and c below that such a g exists and is smooth, although proving it is not difficult.)

Note that sch we have only given a geomoetric definition of orientation as a "direction" and not any sort of formula, what you should give in resonse is some sort of geometric justification.

b) Show that if we have smooth maps $f_{12} : C_1 \to C_2$, $f_{23} : C_2 \to C_3$, $f_{13} = f_{23} \circ f_{12}$, then for any $\omega \in \Omega(C_3)$, we have $f_{13}^*(\omega) = f_{12}^*(f^{23}(\omega))$. (Note: this is **not** simply the defininition of composition of functions. You need to unpack the definitions and do a little work.)

c)Let $\alpha : [a, b] \to C, \alpha[a', b'] \to C$ be two parametrization of C which define the same orientation. Show that for any 1-form ω on V we have $\int_a^b \alpha^{*,1}\omega = \int_{a'}^{b'} \alpha'^{*,1}\omega$ (Hint: use part b and the lemma shown in class.)

5. Let $C_1 = \alpha(t) = (t, t), C_2 = \beta(s) = (s, s^2)$ and C_3 be a curve which consists of two legs of the triangle with verticies at $(0, 1), (0, 0)$ and $(1, 1)$ oriented to begin at $(0, 0)$ and end at $(1, 1)$.

a) Find the integrals $\int_{C_i} \omega$ where $\omega = (6x - y^2)dx - 2xydy$

b) The result may surprise you. Find a reason for such an answer.

6.Let C be the curve in $\mathbb{R}^3$ parametrized by the map $\alpha : [-1, 2] \to \mathbb{R}^3, (\alpha t) := (\cos \pi t, \sin \pi t, t)$. Evaluate

$$\int_C (1 + x)dx - xydy + (y + z)dz$$

7. Which of the following statements are true

a)If C is a vertical line segment, then $\int_C p(x, y)dx = 0$ for any smooth function p on $\mathbb{R}^2$.

b)If p, q are smooth positive functions on $\mathbb{R}^2$ then for any smooth curve C we have $\int_C pdx + qdy \geq 0$

8. Fix a number $r, 0 < r < \pi$. Let $C \subset \mathbb{R}^3$ be given by a pair of equations

$$x^2 + y^2 + z^2 = a^2, y = xtgr$$

. a) Show that C is a smooth curve and find a parametrization α : $[0, 1] \to C$ of C such that we go along C in the counter clock direction [if one looks from the side of positive x.]

b) Find $\int_C (y - z)dx + (z - x)dy + (x - y)dz$