I'll start with some definitions and results which we learned in the end of Fall. The problems 1 and 2 are from the homework for December 20-th.

Let V be a vector space of dimension d. For any $r \geq 0$ we denote by $\Lambda^r(V^\vee)$ the space of antisymmetric r-linear forms on V

[that is elements of ω of $\Lambda^r(V^\vee)$ are functions $\omega(v_1, v_2, ..., v_r), v_i \in V, 1 \leq i \leq r$ on V which are linear in any argument and such that for and permutations $\sigma \in S_r$ we have

$\omega(v_{\sigma(1)}, v_{\sigma(2)}, ..., v_{\sigma(r)}) = \epsilon(\sigma)\omega(v_1, v_2, ..., v_r)$]

Remark. By the definition $\Lambda^1(V^\vee) = V^\vee$ and $\Lambda^2(V^\vee) = \mathcal{B}_{as}(V)$. We define $\Lambda^0(V^\vee) := \mathbb{R}$.

Let $\mathcal{I}(r, d)$ be the set of all sequences $I = (i_1, ...i_r), 1 \leq i_1 < ... < i_r \leq d$.

1. Find the number of elements in the set $\mathcal{I}(r, d)$.

2. a) Fix a basis $\{e_1, ..., e_d\}$ of V. For any $I = (i_1, i_2) \in \mathcal{I}(2, d)$ we define a bilinear form $\omega^I \in \Lambda^2(V^\vee)$ by

$$[*]\omega^I(v_1, v_2) := e^1(v_1)e^2(v_2) - e^1(v_2)e^2(v_1)$$

.

Show that ω^I is the unique antisymmetric bilinear form on V such that

$\omega^I(e_{i_1}, e_{i_2}) = 1$ and for any $p_1, p_2, 1 \leq p_1, p_2 \leq d$ such that $\omega^I(e_{p_1}, e_{p_2}) \neq 0$ we either have $p_1 = i_1, p_2 = i_2$ or $p_1 = i_2, p_2 = i_1$

b)Prove that the set $\omega^I, I =\in \mathcal{I}(2, d)$ is a basis of the space $\Lambda^2(V^\vee)$.

c) Show that for any $I = (i_1, i_2, i_3) \in \mathcal{I}(3, d)$ there exists unique bilinear form $\omega^I \in \Lambda^3(V^\vee)$ such that

$\omega^I(e_{i_1}, e_{i_2}, e_{i_3}) = 1$ and for any $p_1, p_2, p_3, 1 \leq p_1, p_2, p_3 \leq d$ such that $\omega^I(e_{p_1}, e_{p_2}, e_{p_3}) \neq 0$ there exists a permutations $\sigma \in S_3$ such that $p_k = i_{\sigma(k)} 1 \leq k \leq 3$. Write an explicit formula for $\omega^I(v_1, v_2, v_3), v_k \in V$ similar to the formula $*$ in a).

d) Show that $\omega^I, I =\in \mathcal{I}(3, d)$ is a basis in the space $\Lambda^2(V^\vee)$ and find the dimension of the space $\Lambda^3(V^\vee)$.

e)generalize the results of d) to the case of an arbitrary r.

Given any $\omega' \in \Lambda^{r'}(V^\vee), \omega'' \in \Lambda^{r''}(V^\vee)$ we define a function $\omega(v_1, ..., v_r), r = r' + r''$ by the rule

$$\omega(v_1, ..., v_r) := \sum_{\sigma \in S_r} \epsilon(\sigma)\omega'(v_{\sigma(1)}, ..., v_{\sigma(r')})\omega''(v_{\sigma(r'+1)}, ..., v_{\sigma(r'+r'')})/r'!r''!$$

We denote this function $\omega(v_1, ..., v_r)$ by $\omega' \wedge \omega''$ Example. If r'=r"=1 we have

$$\omega' \wedge \omega''(v_1, v_2) = \omega'(v_1)\omega''(v_2) - \omega'(v_2)\omega''(v_1)$$

3. a) Show that $\omega' \wedge \omega'' in \Lambda^r$ [that is show that the function $\omega' \wedge \omega''(v_1, ..., v_r)$ is linear in all argument and that for and permutations $\sigma \in S_r$ we have

$\omega(v_{\sigma(1)}, v_{\sigma(2)}, ..., v_{\sigma(r)}) = \epsilon(\sigma)\omega(v_1, v_2, ..., v_r)$]

b)Show that for any $I' = (i'_1, ...i'_{r'}) \in \mathcal{I}(r', d), I'' = (i''_1, ...i''_{r''}) \in \mathcal{I}(r'', d)$ we have $\omega^{I'} \wedge \omega^{I''} = 0$ if the intersection $I' \cap I'' \neq \emptyset$.

c) Show that for any $I' \in \mathcal{I}(r', d), I'' \in \mathcal{I}(r'', d)$ such that $I' \cap I'' = \emptyset$ we have $\omega^{I'} \wedge \omega^{I''} = \pm\omega^I, I := I' \cup I''$. and $\pm = (-1)^{c(I',I'')}$ where $c(I', I'')$ is the number of pairs $(p, q), 1 \leq p \leq r', 1 \leq q \leq r''$ such that $i'_p > i''_q$

If it is difficult to give a prove in general consider the cases when $(r', r'') = (1, 1), (r', r'') = (1, 2)$ and $(r', r'') = (2, 1)$

d) Solve problems from Chapter 15 Section 1 Number 2

Section 2 Numbers 3,4

4)Chapter 14 Section 1 Numbers 1,3 6

Let $\Sigma = \{(x_1, x_2, x_3)|x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$.

For any $\sigma \in \Sigma$ we denote by $T_\Sigma(\sigma) \subset \mathbb{R}^3$ the tangent space to Σ at σ.

Let $\omega \in \Omega^2(\mathbb{R}^3) := x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2$. By the definition for any $v = (x_1, x_2, x_3) \in \mathbb{R}^3, \omega(v)$ is a bilinear form on $\mathbb{R}^3$ such that $\omega(v)(e_1, e_2) = x_3, \omega(v)(e_1, e_3) = -x_2, \omega(v)(e_2, e_3) = x_1$. For any $\sigma \in \Sigma$ we denote by $\omega_\Sigma(\sigma) \in \mathcal{B}_{as}(T_\Sigma(\sigma))$ the restriction of the form $\omega(\sigma)$ on the subspace $T_\Sigma(\sigma)$

a) Show that $\omega_\Sigma(\sigma) \neq 0$ for all $\sigma \in \Sigma$

Since $\omega_\Sigma(\sigma) \neq 0$ the form $\omega_\Sigma(\sigma) \in \mathcal{B}_{as}(T_\Sigma(\sigma))$ defines an orientation on the tangent space $T_\Sigma(\sigma)$

b) Check whether this orientation agrees with the orinetation on $T_\Sigma(\sigma)$ given by counterclockwise orientation if you look from inside.

A hint. Construct a basis f_1, f_2 of $T_\Sigma(\sigma)$ such that the rotation from f_1 to f_2 goes into the counterclockwise direction [if you look from inside] and check whether the number $\omega_\Sigma(\sigma)(f_1, f_2)$ is positive.

A second hint. It is difficult to construct explicitely a basis f_1, f_2 of $T_\Sigma(\sigma)$ for all $\sigma = (x_1, x_2, x_3) \in \Sigma$. So construct such basis separately for three cases when $x_i \neq 0, i = 1, 2, 3$