Both Kahzdan's first and second problem require the following lemma: let $f : [a,b] \to [c,d]$ be a continuous either strictly increasing or strictly decreasing function with $f(a) = c$, $f(b) = d$ or vice versa. Then f is invertible.

Proof: We show this for the case of f increasing, the decreasing case is almost precisely identical. Let $c \leq y \leq d$. By the intermediate value theorem, there is x, $a < x < b$ such that $f(x) = y$. We assume for contradiction there are two values, $x_1 \neq x_2$ such that $f(x_1) = f(x_2) = y$. But then either $x_1 < x_2$ or $x_1 > x_2$. In the first case, $f(x_1) < f(x_2)$ and in the second case $f(x_1) > f(x_2)$, a contradiction either way.

Kahzdan 1) By the mean value theorem, $(f(x_0 + a) - f(x_0))/a = f(c)$ for some $c \in [x_0, x_0 + a]$ so $(f(x_0 + a) - y_0)/a > C$. Rearranging this we get $f(x_0 + a) > y_0 + aC$. Similarly, $f(x_0 - a) < y_0 - aC$. So $[y_0 - aC, y_0 + aC] \subset [f(x_0 - a), f(x_0 + a)]$, so if $|y - y_0| < aC$. Now we claim f is increasing, so f is invertible on $[f(x_0 - a), f(x_0 + a)]$ and thus on any subset of it. To show f is increasing, take $x_1 < x_2$. Then $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > C(x_2 - x_1) > 0$ where c is some value in $[x_1, x_2]$.

Kahzdan 2) a) By the remarks above problem 1, g exists.

Now, we must show g is continuous. We will show continuity at the interior points of (α, β), the case of the endpoints is almost identical. Let $x \in (a,b)$, $\epsilon > 0$. We may assume ϵ is small enough that $(x-\epsilon, x+\epsilon) \subset (a,b)$. Then we know $f(x-\epsilon) > f(x) > f(x+\epsilon)$, take δ small enough that $(f(x)-\delta, f(x)+\delta) \subset (f(x+\epsilon), f(x-\epsilon))$. Then we claim that if $|y - f(x)| < \delta$ then $|g(y) - g(f(x))| = |g(y) - x)| < \epsilon$. This will establish continuity at $f(x)$, since every point of (α, β) is of the form $f(x)$ for some x, this will establish the continuity of g on (α, β).

By our choice of δ, $|y - f(x)| < \delta$ implies $f(x+\epsilon) < y < f(x-\epsilon)$ so $x - \epsilon < g(y) < x + \epsilon$. So $|g(y) - x| < \epsilon$.

b) If $f' \neq 0$ then the intermediate value theorem guarantees us g is continuously differentiable. Conversely, if g is continuously differentiable, then differentiating $g(f(x)) = x$ gives $g'(f(x))f'(x) = 1$ by the chain rule so $f'(x) \neq 0$. As $f'(x)$ is assumed continuous, this implies it is either always > 0 or always < 0. By the Mean Value Theorem, there is some $c \in (a,b)$ with $f'(c) = (f(b) - f(a))/(b - a) < 0$ so < 0 is the correct choice.

3) By the Implicit Function Theorem, we just need to verify that either $\partial f \partial x$ or $\partial f/\partial y$ is nonzero at $(0,0)$. We are told $D_f \neq 0$, and in explicit coordinates, D_f is the map $(x,y) \to (\partial f/\partial x)x + (\partial f/\partial y)y$, so we can not have both coefficents 0.

9.1.3) It is easy to see f can have no gloabl inverse: $f(1,0) = f(1, 2\pi) = (1,0)$. To show that f has a local inverse, we just need to check the conditions of the inverse function theorem, the only non trivial one is that D_f is invertible. The matrix of D_f is

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

so its determinant is $r\cos^2\theta + r\sin^2\theta = r > 0$.

Note: If we extend the map to $r = 0$ by defining $f(0,\theta) = (0,0)$, f is still continuously differnetiable, but it is not locally invertible at $(0,0)$. It is instructive to see how to show this rigorously. We set $D_s = \{(x,y)|x^2 + y^2 < s\}$ Assume for contradiction there exists $s > 0$ and a function $g : D_s \to \mathbf{R}^2$ with $f(g(x,y)) = (x,y)$. We will sometimes write $g(x,y) = (r(x,y), \theta(x,y))$. Let $\theta(0,0) = \theta_0$, clearly $r(0,0) = 0$. By the continuity of g, we may take $0 < t < s$ such that $\theta(D_t) \in (\theta_0 - pi/2, \theta_0 + \pi/2)$. But then it is impossible for $f(g(-(t/2)\cos\theta, -(t/2)\sin\theta)) = (-(t/2)\cos\theta, (t/2)\sin\theta)$ as it should. (Details of this computation left to the reader.)

9.1.4) Each of these functions except d) is continuously differentiable, so we can apply problem 2b) above and conclude that the interval in question is the largest one on which f' has a constant sign. Moreover, as f' is continuous, this is the same as the largest interval on which f' is not 0. For each function we first solve for when $f' = 0$.

a) $f'(x) = 2x - 7 = 0$, $x = 7/2$. The interval is $(\infty, 7/2)$.

b) $f'(x) = 3x^2 - 10x + 3 = (3x-1)(x-3) = 0$, $x = 1/3, 3$. The interval is $(-\infty, 1/3)$.
c) $f'(x) = \cos x = 0$, $x = \ldots - 3\pi/2, -\pi/2, \pi/2, 3\pi/2, \ldots$. The interval is $(\pi/2, 3\pi/2)$.
d) $f'(x) = \cos x = 0$, $x = \ldots - 3\pi/2, -\pi/2, \pi/2, 3\pi/2, \ldots$. The interval is $(-\pi/2, \pi/2)$.
e) This is the function which is not continuously differentiable, in fact it is not even continuous! On the interval $(-pi/2, \pi/2)$, $f'(x) = \sec^2 x > 0$, so when restricted to this interval f has a continuous inverse. If we extend the interval any further, f will contain both t and $t+\pi$ for some t. As $\tan t = \tan(t+\pi)$, f is not injective on any larger interval and has no chance of having any inverse, continuosly differnetiable or not.
f) $f'(x) = e^x \neq 0$ for any x. This function is invertable on the entire interval $(-\infty, \infty)$.
g) $f'(x) = 2xe^{x^2} = 0$, $x = 0$. The interval is $(0, \infty)$.
h) $f'(x) = \cos x - \sin x = 0 \Leftarrow \sin x = \cos x \Leftarrow \tan x = 1$ (we may divide by $\cos x$ because $\sin^2 x + \cos^2 x = 1$ makes it impossible for $\cos x$ and $\sin x$ to be equal if one of them is 0.) So $x = \ldots - 3\pi/4, \pi/4, 5\pi/4 \ldots$. The interval is $(-3\pi/4, \pi/4)$.

9.1.5) In every case, the formula is $1/f'(x_0)$.
a) $-1/5$
b) $1/3$
c) $1/\cos 3$
d) $1/\cos 1$
e) 1
f) $1/e^2$
g) $-1/(2e)$
h) 1

9.4.1) In each case, the idea is to call the left hand side of the equation F and apply the implicit function theorem to get $dy/dx = (\partial F/\partial x)/(\partial F/\partial y)$.
a) $\partial F/\partial x = 2x + y$, $\partial F/\partial y = x + 3y^2$, $dy/dx = 4/13$.
b) $\partial F/\partial x = \cos x$, $\partial F/\partial y = -\sin y$, $dy/dx = 0/(-1) = 0$.
c) $\partial F/\partial x = 2x$, $\partial F/\partial y = 3y^2$, $dy/dx = -4/12 = -1/3$.
d) $\partial F/\partial x = e^x$, $\partial F/\partial y = \sec^2 y$, $dy/dx = 1/1 = 1$.
e) $\partial F/\partial x = y - 2x$, $\partial F/\partial y = x + y^5$, $dy/dx = 0/3 = 0$.

9.4.2) In this case, dx/dy is not always defined. It is given by $1/(dy/dx)$ when $dy/dx \neq 0$, by the inverse function theorem, and is not defined when $dy/dx = 0$ by Problem 2.b. We have
a) 13/4
b) undefined
c) -3
d) 1
e) undefined

9.4.5) a) x and y are related by $F(x,y) = 0$ where $F(x,y) = (x^2+y^2)^2 - 2a^2(x^2 - y^2)$, so by the implicit function theorem, x can be expressed as a function of y or vice versa unless $\partial F/\partial x = \partial F/\partial y = 0$. We first determine what these bad points are.

$\partial F/\partial y = 4x(x^2+y^2) + 4a^2 y = 4y(x^2+y^2+a^2)$. This equals 0 only if $y = 0$. If $y = 0$, then $x^4 = 2a^2x^2$ so $x = 0$ or $x = \pm\sqrt{2}a$. In the second case,it is easy to check $partial F/\partial x \neq 0$. So We can locally express y as a function of x at all points other than $(\pm\sqrt{2}a, 0)$ and $(0,0)$, and at the former we can express x as a function of y. We now show there is no neighborhood of $(0,0)$ where x can be expressed as a function of y or *vice versa*.

Let's get an inuitive sense of what this set looks like. For x and y small, the quadratic terms should over power the fourth powers, so the set should be roughly the same for small x and $y$as the zero set of $x^2 - y^2 = (x-y)(x+y)$, which is simply the two crossing lines $x = y$ and $-x = y$. More rigorously, consider any vector (u,v) with $u^2 - v^2 \neq 0$. We show that for sufficently small r, $F(ru, rv)$ has the same sign as $v^2 - u^2$. This is easy, we want to show $v^2 - u^2$ and $2a^2r^2(v^2 - u^2) + r^4(u^2+v^2)^2$ have the same sign, as

$v^2 - u^2$ and $2a^2r^2(v^2 - u^2)$ have the same sign, we just need to show $(u^2 + v^2)^2r^4 < |2a^2(v^2 - u^2)|r^2$. This is indeed true, just take $r^2 < |2a^2(v^2 - u^2)/(u^2 + v^2)^2|$.

It is not to hard at this point to show the lemnicaste is, in a neighborhood of $(0, 0)$, the union of two smooth curves, one with derivative 1 and one with derivative -1. To show that neither x nor y is a function of the other, however, is easier. Simply take a δ such that F is positive at $(\pm 2r, \pm r)$ and negative at $(\pm r, \pm 2r)$ for $r < \delta$. (Where all combinations of $\pm$ are considered.) For any x with $\sqrt{5}|x| < r$, F is positive at $(x, 2x)$ and $(x, -2x)$ and negative at $(x, x/2)$ and $(x, -x/2)$, so it must have a zero (x, y) with $y \in (x/2, x)$ and one with $y \in (-x, -x/2)$. Thus, y is not a function of x on any neighborhood of 0. Similarly, x is not a function of y on any neighborhood of 0.