1a) For any function g, the meaning of $dg(v)(v)$ is $\lim_{\epsilon\to 0}(g(v+\epsilon v)-g(v))/\epsilon$. In this case, this is $\int\omega/\epsilon$ where the integral is along the line segment from v to $(1+\epsilon)v$. Parameterizing this by $\alpha:[1,1+\epsilon]\to V$, $t\to tv$, we have $\alpha^*\omega(t)(e_1)=\omega(tv)(v)$ where e_1 is the unit vector in the t direction. So we have to compute

$$\lim_{\epsilon\to 0}\int_1^{1+\epsilon}\omega(tv)(v)dt/\epsilon.$$

By the fundamental theorem of calculus, this is $\omega(1\cdot v)(v)=\omega(v)(v)$.

b) We will compute $df_\omega(v)(w)$. This expression can be equivalently thought of as $(d/ds)f_\omega(v+sw)$. Let γ_1 be the line segment from 0 to v, γ_2 the line segment from 0 to $v+sw$ and γ_3 the line segment from v to $v+sw$, where in each case the line segment is give the orientation with the endpoints in the described order. So what we are being asked to compute is

$$\lim_{s\to 0}(\int_{\gamma_2}\omega-\int_{\gamma_1}\omega)/s.$$

At this point, we use Stoke's theorem. Let Δ be the triangle bounded by 0, v and $v+s\omega$. By Stoke's theorem

$$\int_\Delta d\omega=\int_{\gamma_1}\omega+\int_{gamma_3}\omega-\int_{\gamma_2}\omega.$$

But $d\omega=0$, so this integral is 0 and

$$\int_{\gamma_2}\omega-\int_{\gamma_1}\omega=\int_{\gamma_3}\omega.$$

(This is one of the most frequent ways Stoke's theorem is used. Essentially, if a one form has $d\omega=0$, we can integrate it along any path we like, and the answer will only depend on the endpoints.) So we need to compute

$$\lim_{s\to 0}\int_{\gamma_3}\omega/s$$

.

As in part a, this turns out to be

$$\lim_{s\to 0}\int_0^s\omega(v+tw)(w)dt$$

or $\omega(v)(w)$ by the fundamental theorem of calculus.

c) The idea of the proof will be the same as above, with u_0 taking the role of 0. The key point will be to make sure that all integrals are over regions that actually lie in U, as they would otherwise be nonsense.

We define f_ω by $f_\omega(v)=\int_\gamma\omega$ where γ is the line segment from u_0 to v that is assumed to exist. We compute $df_\omega(v)(w)$, as before, this is $(d/ds)|_{s=0}f_\omega(v+sw)$. As U is open, for s small enough, $v+s'w$ lies in U for all $s'\leq s$, from now on, let s be this small. Let γ_1 be the path from u_0 to v, γ_2 from v to $v+sw$ and γ_3 from u_0 to $v+sw$. Let Δ be the triangle bounded by the γ's.

By assumption on the smallness of s, γ_2 lies in U. γ_1 and γ_3 are then lien segments of the sort assumed to lie in U and Δ is a union of such line segments, so all of these sets lie in U. Thus, integrals over them are well defined and the proof can proceed as before.

d) This is probably obvious, but note that the plane minus a single point does not obey the geometric condition of part c, which is why this example is possible. The key point is that the integral on Δ no longer makes sense. We do not actually need the full geometric condition of c, what is crucial is that any closed path in U be the boundary of a surface in U.

Checking $d\omega=0$ is easy. We have

$$d\omega=(\frac{d}{dy}\frac{-y}{x^2+y^2}-\frac{d}{dx}\frac{x}{x^2+y^2})dx\wedge dy$$

$$= (\frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2})dx \wedge dy$$

$$= (\frac{-2}{x^2+y^2} + \frac{2(x^2+y^2)}{(x^2+y^2)^2})dx \wedge dy = 0.$$

Now, if $\omega = df$ then for any map $\gamma : [0,1] \to U$, we'd have $\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))$. (I am slightly abusing notation by using γ as both the path itself and it's paramteerization, but I don't think this should be confusing.) In particular, let $\gamma : [0, 2\pi] \to U$, $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$. As $\gamma(0) = \gamma(1)$, we should have $\int_\gamma \omega = 0$.

In fact, we have

$$\gamma^*(t) = \frac{-\sin t}{\sin^2 t + \cos^2 t}(-\sin t dt) + \frac{\cos t}{\sin^2 t + \cos^2 t}(\cos t dt) = dt.$$

So $\int_\gamma \omega = \int_0^{2\pi} \gamma^* \omega = \int_0^2 \pi dt = 2\pi$.

2) Let $\omega = \sum f dx_{k_1} \wedge \ldots \wedge dx_{k_i}$. Then $d^2\omega = \sum (d^2 f) dx_{k_1} \wedge \ldots \wedge dx_{k_i}$. So it suffices to prove that $d^2 f = 0$ for any $f \in \Omega^0$. This was shown in class.

3) Write $\omega' = \sum_{I'} f'_{I'} \omega_{I'}$, $\omega'' = \sum_{I''} f''_{I''} \omega_{I''}$ where the $\omega_{I'}$ and $\omega_{I''}$ are the bases of $\Omega(i')$ and $\Omega(i'')$ introduced in the previous problem set and the $f'_{I'}$ and $f''_{I''}$ are 0-forms. As $\wedge$ is linear in both terms and the f's are just real valued functions, we have

$$\omega' \wedge \omega'' = \sum_{I'} \sum_{I''} f'_{I'} f''_{I''} \omega_{I'} \wedge \omega_{I''}$$

and

$$d(\omega' \wedge \omega'') = \sum_{I'} \sum_{I''} d(f'_{I'} f''_{I''}) \omega_{I'} \wedge \omega_{I''}$$

$$\sum_{I'} \sum_{I''} (df'_{I'} f''_{I''} + f'_{I'} df''_{I''}) \omega_{I'} \wedge \omega_{I''}.$$

It is obvious that these will correspond to the two terms in the final result, the point is simply to get the sign right. The left terms of each summand combine to give

$$\sum_{I'} \sum_{I''} df'_{I'} f''_{I''} \omega_{I'} \wedge \omega_{I''}.$$

As $f''_{I''}$ is merely a real valued function, it commutes with the other terms and we get

$$\sum_{I'} \sum_{I''} df'_{I'} \omega_{I'} \wedge f''_{I''} \omega_{I''}.$$

As $\wedge$ is distributive, we get

$$(\sum_{I'} df'_{I'} \omega_{I'}) \wedge (\sum_{I''} f''_{I''} \omega_{I''}) = d\omega' \wedge \omega''$$

Now for the second sum. The key point is that, as $df''_{I''}$ is a one form, we can not simply write $df''_{I''} \wedge \omega_{I'} = \omega_{I'} \wedge df''_{I''}$. There will be sokme change of sign, and we need to work out what it is. $\omega_{I'}$ constists of a wedge of the form $dx_{i_1} \wedge dx_{i_2} \ldots \wedge dx_{i_{i'}}$. We can thinking of moving $df''_{I''}$ to the end as switiching it in turn with each $d_{x_{i_k}}$. Each of these switches will introduce a factor of (-1) and there are i' swithced, so we get $df''_{I''} \wedge \omega_{I'} = (-1)^{i'} \omega_{I'} \wedge df''_{I''}$. Using this in the sum, we have

$$\sum_{I'} \sum_{I''} f'_{I'} df''_{I''} \wedge \omega_{I''} \wedge \omega_{I''} = (-1)^{i'} \sum_{I'} \sum_{I''} f'_{I'} \omega_{I''} \wedge df''_{I''} \wedge \omega_{I''}$$

$$= (-1)^{i'} (\sum_{I'} f'_{I'} \omega_{I'}) \wedge (\sum_{I''} df''_{I''} \wedge \omega_{I''}) = (-1)^{i'} \omega' \wedge \wedge''$$

Combining these two results, we get the claim.

4) We will prove this result for ω a 0-form, for ω of the form dx, x a 0-form, and for ω a wedge of two form for which the result has already been proven. Every form is a some of forms that can be built through these steps, this will prove the claim.

The 0-form. Write $\omega = f$, for f a real valued function. Then

$$\phi^*(df) = D_f \circ D_\phi$$

and

$$d(\phi^*(f)) = D_f \circ \phi$$

so this is just the chain rule.

The form dx. We have

$$\phi^*(d(dx)) = \phi^*(0) = 0$$

. But we have just shown above that $\phi^*(dx) = d\phi^*(x)$, so

$$d(\phi^*(dx)) = d(d\phi^*(x)) = 0$$

.

The wedge product. We first show that $\phi^*(\omega_1 \wedge \omega_2) = \phi^*(\omega_i) \wedge \phi^*(\omega_2)$. This is really trivial, let $v_1, \ldots, v_{i_1}, v_{i_1+1} \ldots, v_{i_1+i_2}$ be the vectors to which we are going to apply these two expressions, at the point x. The left hand side is, by definition

$$(\omega_1 \wedge \omega_2)(\phi(x))(D_\phi(v_1), \ldots, D_\phi(v_{i_1+i_2}))$$

Which is a sum over all permuations σ of

$$\omega_1(D_\phi v_{\sigma(1)}, \ldots, D_\phi v_{\sigma(i_1)})\omega_2(D_\phi v_{\sigma(i_1+1)}, \ldots, D_\phi v_{\sigma(i_1+i_2)})$$

with certain coefficents, wherer the forms are evaluated at $\phi(x)$. The other expresison we are to evaluate is the sum with the same coefficents of

$$\phi^*(\omega_1)(v_{\sigma(1)}, \ldots, v_{\sigma(i_1)})\phi^*(\omega_2)(v_{\sigma(i_1+1)}, \ldots, v_{\sigma(i_1+i_2)})$$

But the definitions of these ϕ^*'s are precisely the expressions in the first computation.

We now proceed to prove that $d(\phi^*(\omega_1 \wedge^\omega_2)) = \phi^*(d(\omega_1 \wedge \omega_2))$. By the above, the left is $d(\phi^*(\omega_1) \wedge \phi(\omega_2))$. Or

$$d(\phi^*(\omega_1)) \wedge \phi^*(\omega_2) + (-1)^{i_1}\phi^*(\omega_1) \wedge d\phi^*(\omega_2).$$

The right hand side is

$$\phi^*(d\omega_1 \wedge \omega_2 + (-1)^{i_1}\omega_1 \wedge d\omega_2)$$

whichby the above is

$$\phi^*(d\omega_1) \wedge \phi^*(\omega_2) + (-1)^{i_1}\phi^*(\omega_1) \wedge \phi^*(d\omega_2).$$

Since we are assuming we have already proven $d\phi^* = \phi^* d$ for ω_1 and ω_2, we are done.

15.5.3) a) $(2x_1+x_2)(x_1x_3)(x_2-x_3)$ b) $d(x_1x_3) = x_3dx_1+x_1dx_3$ c) $x_1x_3d(2x_1+x_2)-(2x_1+x_2)d(x_2-x_3) = 2x_1x_3dx_1 + (x_1x_3 - 2x_1 - x_2)dx_2 + (2x_1+x_2)dx_3$ d) $d(2x_1+x_2) \wedge d(x_2-x_3) = (2dx_1+dx_2) \wedge (dx_2 - dx_3) = 2dx_1dx_2 - 2dx_1dx_3 - dx_2dx_3$ e) We first work with each term. $(2x_1+x_2)(x_2-x_3)^2 d(2x_1+x_2) \wedge d(x_1x_3) = (2x_1+x_2)(x_2-x_3)^2(2dx_1+dx_2) \wedge (x_1dx_3+x_3dx_1) = (2x_1+x_2)(x_2-x_3)^2(2x_1dx_1dx_3+x_1dx_2dx_3+x_1dx_2dx_3)$ is the first term. $(2x_1+x_2)^3(2dx_1dx_2 - 2dx_1dx_3 - dx_2dx_3)$ is the second, where we have used part d). $\sin(2x_1+x_2)(x_1dx_3+x_3dx_1)(dx_2-dx_3) = \sin(2x_1+x_2)(-x_1dx_2dx_3+x_3dx_1dx_2-x_1dx_1dx_3)$. Adding these together gives the answer. f) $(2dx_1+dx_2)(x_1dx_3+x_3dx_1)(dx_2-dx_3) = 2x_1dx_1dx_3dx_2 - x_3dx_2dx_1dx_3 = (x_3-2x_1)dx_1dx_2dx_3$. g) $((2x_1+x_2)x_1x_3 - (x_2-x_3)^2)(x_3-2x_1)dx_1dx_2dx_3$ where we have used part f) to compute $\phi^*(dx_1dx_2dx_3)$. 15.5.6) a) We computed $\phi^*(\omega)$ above, $d\phi^*(\omega) = d((2x_1+x_2)(x_1x_3)(x_2-x_3)) = (2x_1x_3(x_2-x_3)+(2x_1+x_3)x_3(x_2-x_3))dx_1 + (x_1x_3(x_2-x_3)+(2x_1+x_2)x_1x_3)dx_2 + ((2x_1+x_2)x_1(x_2-x_3)-$

$(2x_1 + x_2)x_1x_3)dx_3$. $d\omega = y_2y_3dy_1 + y_1y_3dy_2 + y_1y_2dy_3$ and $\phi^*d\omega = (x_1x_3)(x_2 - x_3)(2dx_1 + dx_2) + (2x_1 + x_2)(x_2 - x_3)(x_1dx_3 + x_3dx_1) + (2x_1 + x_2)(x_1x_3)(dx_2 - dx_3)$. It is easy to compare terms and see that these are equal. b)$d\omega = 0$, so we just need to show that $d\phi^*\omega = 0$. This is $d(x_3dx_1 + x_1dx_3) = dx_3dx_1 + dx_1dx_3 = 0$. c)$d\omega = dy_2dy_1 - dy_1dy_3$. $\phi^*d\omega = d(2x_1 + x_2)(-d(x_1x_3) - d(x_2 - x_3)) = (2dx_1 + dx_2)(-x_1dx_3 - x_3dx_1 - dx_2 + dx_3) = -2x_1dx_1dx_3 - 2dx_1dx_2 + 2dx_1dx_3 - x_1dx_2dx_3 + x_3dx_1dx_2 + dx_2dx_3 = (x_3 - 2)dx_1dx_2 + (2 - 2x_1)dx_1dx_3 + (1 - x_1)dx_2dx_3$. Now $d\phi^*\omega = d(2x_1x_3dx_1 + (x_1x_3 - 2x_1 - x_2)dx_2 + (2x_1 + x_2)dx_3) = -2x_1dx_1dx_3 + x_3dx_1dx_2 - x_1dx_2dx_3 - 2dx_1dx_2 + 2dx_1dx_3 + dx_2dx_3 = (x_3 - 2)dx_1dx_2 + (2 - 2x_1)dx_1dx_3 + (1 - x_1)dx_2dx_3$. d) $d\omega = 0$ so our job is to show $d\phi^*\omega = 0$. This is $d(2dx_1dx_2 - 2dx_1dx_3 - dx_2dx_3)$ and sure enough, every term is 0. 16.4.1) Choose the orientation on $\mathbf{RR}^3$ given by $dxdydz$. We will use Stoke's theorem, so we must first compute $d(x^2dydz + y^2dzdx + z^2dxdy)$ This is $2xdxdydz + 2ydydzdx + 2zdzdydx = 2(x + y + z)dxdydz$. Now, this is the integral of an odd function over a region symmetirc about the origin, hence 0. 16.4.3) We just need to show one of these formulae is correct, we pick $Vol(U) = \int_{\partial U} xdydz$. Since $Vol(U)$ is defined by $\int_U dxdydz$ it suffices by Stoke's theorem to show that $d(xdydz) = dzdydz$. But this is easy. 16.4.5) We first prove the first identity. Consider the two form $\omega = f((\partial g/\partial x)dydz + (\partial g/\partial y)dzdx + (\partial g/\partial z)dxdy)$. The integral of this over ∂U is precisely the right hand side. So by Stoke's theorem, this is the same as $\int_U d\omega$. We now compute $d\omega$. We get

$$df \wedge ((\partial g/\partial x)dydz + (\partial g/\partial y)dzdx + (\partial g/\partial z)dxdy) + fd(((\partial g/\partial x)dydz + (\partial g/\partial y)dzdx + (\partial g/\partial z)dxdy))$$

The first term is

$$((\partial f/\partial x)dx + (\partial f/\partial y)dy + (\partial f/\partial z)dz) \wedge ((\partial g/\partial x)dydz + (\partial g/\partial y)dzdx + (\partial g/\partial z)dxdy)$$

Expanding this out, only three terms are nonzero, these terms give

$$((\partial f/\partial x)(\partial g/\partial x) + (\partial f/\partial y)(\partial g/\partial y) + (\partial f/\partial z)(\partial g/\partial z))dxdydz$$

which is precisely $< \nabla f, \nabla g >$. The second term,

$$fd((\partial g/\partial x)dydz + (\partial g/\partial y)dzdx + (\partial g/\partial z)dxdy)$$

can be computed similarly; the only nonzero terms give

$$f((\partial^2 g/\partial x^2) + (\partial^2 g/\partial y^2) + (\partial^2 g/\partial z^2))dxdydz$$

which is $f \triangle g$.

To prove the second identity, apply the first identity to f and g and then to g and f, subtract the results.