1) One thing neither professor Kazhdan or I caught until it was too late to be worth changing is that U was used to mean two different things in defintion Q1. While I doubt this confused anyone, it made the problem incredibly annoying to write up. In this problem set, we will refer to the set in the main part of definition Q1 as U and the set in part (c) as V.

a) That Q1 implies Q2 is easy. Take $F(x^1, \ldots, x^n) = x^i - f(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n)$. Let U be the set V in defnition Q1. Checking that this works is straightforward.

The reverse direction is almost exactly the content of the of the Implicit Function Theorem. We just have to show there exists and i such that D_f restricted to the x^i axis is invertible, i.e. (since it is one dimensional) nonero. Since $D_f \neq 0$ and the x^i are a basis, some such i must exist.

b) We claim we can take U to be the U of $Q1$ and $\phi(x^1, \ldots, x^n) = (x^1, \ldots, x^{i-1}, f(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n), x^{i+1}, \ldots, x^n)$ Finally, since the V of Q1 is assumed open, there is an r_0 such that $B_{\sigma_0}(r_0) \subset V$, we take this to be r_0.

Part (a) of Q3 is automatic. Since $\phi(U) \cap V = \Sigma \cap V$ and $B_{\sigma_0}(r_0) \subset V$, we have $\phi(U) \cap B_{\sigma_0}(r_0) = \Sigma \cap B_{\sigma_0}(r_0)$, which is part (c). To see that D_f is injective, we just need to show that there is a linear map P such that $P \circ D_f = 1$. This is because, if $D_f v = 0$, then $v = 1v = P \circ D_f v = P0 = 0$. We claim such a map is the projection onto the plane spanned by x^1, x^2, $\ldots$ x^{i-1}, x^{i+1}, $\ldots$, x^n. $P \circ D_f = D_{P \circ f}$ and $P \circ f$ is by definition the identity map.

This only leaves part (d). Since U' is open and $0 \in U'$, there is an $r_0' < r_0$ such that $B_0(r_0') \subset U'$. We claim for $r < r_0'$, $\Sigma \cap B_{\sigma_0}(r) = \phi(U') \cap B_{\sigma_0}(r)$. As $\phi(U') \subseteq \phi(U)$, $\phi(U) \cap B_{\sigma_0}(r_0) = \Sigma \cap B_{\sigma_0}(r_0)$ and $B_{\sigma_0}(r) \subset B_{\sigma_0}(r_0)$, we get $\phi(U') \cap B_{\sigma_0}(r) \subseteq \Sigma \cap B_{\sigma_0}(r)$. We now must show $\phi(U') \cap B_{\sigma_0}(r) \supseteq \Sigma \cap B_{\sigma_0}(r)$. Let $\sigma \in \Sigma \cap B_{\sigma_0}(r_0')$. As $\Sigma \cap B_{\sigma_0}(r) \subseteq \Sigma \cap B_{\sigma_0}(r_0) = \phi(U) \cap B_{\sigma_0}(r_0) \subseteq \phi(U)$, there is $x \in U$ such that $\sigma = \phi(x)$. We must show $x \in U'$. We know $\sigma \in B_{\sigma_0}(r)$, so $|\sigma - \sigma_0| < r$. Let P be the same is in the previous paragraph. We have $|x - 0| = |P\sigma - P\sigma_0| \leq |\sigma - \sigma_0| < r$ so $x \in B_0(r)$. We chose r_0' such that we would have $B_0(r) \subset U'$, so we are done.

c) Take T to be the embedding $D_\phi(0)$ and p_i such that $p_i \circ T = D_{p_i \circ \phi}(0)$ is invertible. Thus by the inverse function theorem there is an open set $W \subset \mathbf{R}^{n-1}$ and a smooth function $g : U \to \mathbf{R}^{n-1}$ such that $p_i \circ \phi \circ g = \mathrm{Id}$ and $g \circ p_i \circ \phi = \mathrm{Id}$. Let $q_i : \mathbf{R}^n \to \mathbf{R}$ be projection onto the x^i axis. We claim we can take the i of definition Q1 to be i, f to be $q_i \circ \phi \circ g$ and U to be W. Find r_0' such that for $0 < r < r_0'$, $B_{\sigma_0}(r) \cap \Sigma = B_{\sigma_0}(r) \cap \phi(W)$, we take $V = B_{\sigma_0}(r)$ for some $0 < r < r_0'$.

We first need to show, for any $(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) \in U$,

$$(x^1, \ldots, x^{i-1}, f(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n), x^{i+1}, \ldots, x^n) \in \Sigma.$$

Set $(y^1, \ldots y^{i-1}, y^{i+1}, \ldots y^n) = g(x_1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n)$. We claim

$$\phi(y^1, \ldots y^{i-1}, y^{i+1}, \ldots y^n) = (x^1, \ldots, x^{i-1}, f(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n), x^{i+1}, \ldots, x^n).$$

Note that for any $x, x' \in \mathbf{R}^n$, if $p_i x = p_i x'$ and $q_i x = q_i x'$, then $x = x'$.

We first check

$$p_i \circ \phi(y^1, \ldots y^{i-1}, y^{i+1}, \ldots y^n) = p_i(x^1, \ldots, x^{i-1}, f(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n), x^{i+1}, \ldots, x^n).$$

The left hand side is

$$p_i \circ \phi \circ g(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) = \mathrm{Id}(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n),$$

which is trivially the right hand side.

We now show

$$q_i \circ \phi(y^1, \ldots y^{i-1}, y^{i+1}, \ldots y^n) = q_i(x^1, \ldots, x^{i-1}, f(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n), x^{i+1}, \ldots, x^n).$$

The right hand side is $f(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n)$. Plugging in the defnition of f, we get precisely the left hand side.

We now must show that for any $(x^1, \ldots, x^n) \in V \cap \Sigma$, $f(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) = x^i$. (The last paragraph showed our map was into Σ, we are now showing it is onto.) By the choice of V, $(x^1, \ldots x^n) \in \phi(W)$. Say $(x^1, \ldots, x^n) = \phi(w^1, \ldots, w^{i-1}, w^{i+1}, \ldots w^n)$. So

$$(w^1, \ldots, w^{i-1}, w^{i+1}, \ldots w^n) = g \circ p_i(x^1, \ldots x^n) = g(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots x^n).$$

Thus,

$$f(x^1,\dots,x^{i-1},x^{i+1},\dots x^n) = q_i \circ \phi \circ g(x^1,\dots,x^{i-1},x^{i+1},\dots x^n) =$$
$$q_i \circ \phi(w^1,\dots,w^{i-1},w^{i+1},\dots w^n) = q_i(x^1,\dots,x^n) = x^i),$$

as desired.

Note: the key reason that we needed to use part (d) of defnition Q3 was that the inverse function theorem only guaranteed g to be defined on some $W \subset U$, we needed to know that any point of Σ was not only locally the image of some point of U under ϕ but was the image of some point of W, so we could use the function g.

d) Assume the opposite, then, for every i, there exists $v_i \in \mathbf{R}^{n-1}$ such that $v_i \neq 0$, $p_i T v_i = 0$, the latter implies $Tv_i = a_i x^i$ for some a^i. Now, as T is an embedding and the v_i are nonzero, all the Tv_i are nonzero. But then the image of T contains n linearly independent vectors, contradicting that it is the image of a $n-1$ dimensional space.

2) Since f' is continuous and never 0, it is always either < 0 or > 0. This is precisely the situation of the one dimensional inverse function theorem. (Note: this is the difference between the one dimensional and higher dimensional case, only in one dimension can we obtain a global solution.)

3) a) $V = \{(x,y) \neq (0,0)\}$. This is fairly clear, any other point can be written in polar coordinates as (r,θ) for $r > 0$, we can then set $x = \log r$ and $y = \theta$.

b) The linear map is given by the matrix

$$\begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

which has determinant $e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} \neq 0$.

c) $f(0,0) = (1,0) = f(0,2\pi)$.

4) Let us estimate $f(x,y)$, x and y will be constants in what follows. We know there exists $t \in (0,1)$ such that $f(x,y) = x\frac{\partial}{\partial x} f(tx,ty) + y\frac{\partial}{\partial y} f(tx,ty)$. Call this right hand side $h(t)$, note $h(0) = 0$. Applying the mean value theorem to this, we get that there is $u \in (0,t) \subseteq [0,1]$ such that $h(t) = x^2 \frac{\partial}{\partial x} f(ux,uy) + 2xy\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(ux,uy) + y^2 \frac{\partial}{\partial y}^2 f(ux,uy)$. (One form of Taylor's theorem gives precisely what we have just deduced.)

Now, take ϵ such that $\frac{\partial}{\partial x}^2 f(x,y) > 0.9$, $\frac{\partial}{\partial y}^2 f(x,y) > 0.9$ and $|\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y)| < 0.1$ for $|(x,y)| < \epsilon$. Then $|(ux,uy) \leq |(x,y)|$, so we have $f(x,y) > u^2(0.9x^2 - 2 * 0.1|xy| + 0.9y^2)$. We know $u^2 > 0$ as $u \in (0,1)$ and we know $(x,y) \neq (0,0)$, so this will must be positive if the discriminant $4(0.9)(0.9) - (2 * 0.1)^2$ is. It is. (Obviously, the particular values 0.9 and 0.1 could be replaced by others.)