12.3.9) We subdivide this region into 48 pieces, based on which of x, y and z are positive and which negative and on the order of $|x|$, $|y|$ and $|z|$. It is clear each piece has the same volume, we will evaluate the volume of the piece $0 < z < y < x$. This piece is bounded by the preceeding inequalities and $x^2 + y^2 < 1$. Doing the integral on z first, this integral is $\int_0^y dz = y$. We have

$$\int_{0<z<y, x^2+y^2<1} y dx dy.$$

Changing to polar coordinates, we have

$$\int_0^1 \int_0^{\pi/4} (r \sin\theta) r dr d\theta.$$

These are routine integrals, we get $(1^3/3 - 0^3/3)(\cos 0 - \cos(\pi/4)) = (1/3)(1 - 1/\sqrt{2})$. Multiplying by 48, we have $16 - 8\sqrt{2}$. (If you have exceptional visual ability, you can see that you only need to make 12 subdivsions, and the last few are gratuitious. I only figured this out, as some of you saw, in section, long after I'd computed the integral. My approach to problems like this is overcall: make sure you have so many subdivisons that ntohing can be complicated about them.)

12.3.10) I hope people have realized that the volume in question is a pyramid, vertices at $(0, 0, 0)$, $(4, 0, 0)$, $(0, 4, 0)$ and $(0, 0, 2)$.

a) The mass is easy, it's just the volume of the pyrmaid. The area of the base is $(1/2)\dot{4}\dot{4} = 8$ and height is 2 so we have $(1/3)\dot{8}\dot{2} = 16/3$. We show how to compute the z coordinate for he center of mass, the other integrals are almost identical. We first do the integrals on x and y: $\int_{x+y<4-2z} z dx dy$ is just z times the area of an isosceles right triangle with legs of length $4 - 2z$, or $z(1/2)(4 - 2z)^2 = 2z(2 - z)^2$. So the z coordinate of the center of mass is $(\int_0^2 2z(2 - z)^2 dz)/(16/3)$. The integrand is $2z^3 - 8z^2 + 8z$ so the integral is $2(2^4)/4 - 8(2^3)/3 + 8(2^2)/2 = 8 - 64/3 + 16 = 8/3$. Dividing by 16/3, we get 1/2. Doing another such computation to get the x and y coordinates, the center of mass is $(1, 1, 1/2)$

b) Mass: this can be split into two integrals, the integral of x and of z. Notice these are just the integrals we did before to get the center of mass, before we divided through by (16/3), so we know the answers are 16/3 and 8/3 and the mass is $16/3 + 8/3 = 8$. Now to find the center of mass.

OK, remember how I said at dinner I didn't think any of the integrals were too bad? Well, this is the one I hadn't done yet.

The x coordinate: We have $\int_{x+y+2z<4} x^2 + xz dx dy dz$. This is the summ of two terms. Integrating the first one on y and z, we get $\int_{x<4} x^2(4 - x)(2 - x/2)/2 dx = (1/4)\int_{x<4} x^4 - 8x^3 + 16x^2 dx = (1/4)(4^5/5 - 8 * 4^4/4 + 16 * 4^3/3)$. Dividing by 8 and simplifying, $32/5 - 16 + 32/3 = 16/15$. Now, for the second term. Integrating on y, we get $\int_{x+2z<4}(4 - x - 2z)xz dx dz = \int_{z<2}(4 - 2z)(4 - 2z)^2 z/2 - (4 - 2z)^3 z/3 = \int_{z<2}(4 - 2z)^3 z/6 dz = \int_{z<2}(1/2)(4 - 2z)^4/4(1/6) dz = 4^5/(2 * 6 * 4 * 5 * 2) = 32/15$, where we integrated by parts. Dividing by 8, we get 4/15. So the x cooordinate is $16/15 + 4/15 = 4/3$.

The integrals for the y and z coordinates are similar, we get a final answer of (4/3, 4/5, 8/15).

12.3.11) We first find the mass. The integrals on x and y are trivial, so we do them without writing them out to get $\int_0^1 3z\dot{\pi}(1 - z^2) dz = \pi(3/2 - 3/4) = 3\pi/4$. Clearly the x and y coordinates of the center of mass will be 0, the z cooridnate will be $\frac{1}{(3\pi/4)\int_0^1 3z\dot{z}\dot{\pi}(1-z^2)dz=4(1/3-1/5)=8/15.}$

12.3.12) By symmetry, this will lie on the x axis. The integral on x for mass is $\int 64\pi x = (64\pi)(5^2/2)$. SImilarly, the integral of $x\rho$ will be $(64\pi)(5^3/3)$. So the x coordinate is (10/3).

5) The region itself has area 4π, so we just have to subtract off half of the area where the function is negative. There are two regions on which this happens, one above and one below the x-axis. We look at the upper region, the other is its mirror image. We can right it's area as the difference of the areas of two wedges, one bounded by $y = \sqrt{(3)}x$, $y = -\sqrt{(3)}x$ and $x^2 + y^2 = 4$ and the other buonded by the same two lines and $y^2 - x^2 = 2$. We evaluate these by the substitutions $x = r \sin\theta$, $y = r \cos theta$ and $x = r \sinh\tau$, $y = r \cosh\tau$ respectively. The chagne of variables in the second case has Jacobian

$$\begin{pmatrix} \cosh\tau & r \sinh\tau \\ \sinh\tau & r \cosh\tau \end{pmatrix}$$

which has determinant $r(\cosh^2 \tau - \sinh^2 \tau) = r$, the first substitution is classically known to have determinant r. So we get the two integrals

$$\int_{\tan^{-1}(-1/\sqrt(3))}^{\tan^{-1}(1/\sqrt(3))} \int_0^{\sqrt(4)} r dr d\theta$$

and

$$\int_{\tanh^{-1}(-1/\sqrt(3))}^{\tanh^{-1}(1/\sqrt(3))} \int_0^{\sqrt(2)} r dr d\theta$$

These give $4\tan^{-1}(1/\sqrt(3)) = 2\pi/3$ and $2\tanh^{-1}(1/\sqrt(3))$ so we get an area for this wedge of $2(\pi/3 - \tanh^(-1)(1/\sqrt(3))$. The final answer is thus $4\pi - 2\dot{2}\dot{2}(\pi/3 - \tanh^{-1}(1/\sqrt(3))) = 4\pi/3 + 8\tanh^{-1}(1/\sqrt(3))$.

6a) Subtracting the lower two rows from the top one, we have

$$\begin{pmatrix} -2y & -2x & 0 \\ y & x+y & x \\ x+y*x & y & \end{pmatrix}$$

Now adding the half top row to each of the others,

$$\begin{pmatrix} -2y & -2x & 0 \\ 0 & y & x \\ x & 0 & y \end{pmatrix}$$

It is now easy to do an expansion by minors, say on the first row, to get $-2y^3 - 2x^3$.

6b) This maps the space spanned by $(1,0,0,0)$ and $(0,0,1,0)$ to itself by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and the space spanned by $(0,1,0,0)$ and $(0,0,0,1)$ to itself by $\left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right)$. The over all volume change will be the product of the volume changes on each space, so the total volume change will be $(ad - bc)(uy - vx)$. It is easy enough to check thuis by algebra.

7) We have $A^T = -A$, so $\text{Det}A^T = (-1)^n \text{Det}A$. But $\text{Det}A^T = \text{Det}A$ so id n is odd $\text{Det}A = -\text{Det}A$ implies $\text{Det}A = 0$.

8) Following the hint, $A^2 = 4\text{Id}$ (computation omitted because it is easy, but tedious to write out.) So $\text{Det}A^2 = 4^4 = 256$ and $\text{Det}A = \pm 16$. The following is the simplest, although not the easiest to find, arguemnt I can find for how to get the sign. As A is symmetric, we have $A = SDS^{-1}$ where D is a diagonal matrix. So $A^2 = (sDs^{-1})(SDS^{-1}) = SD^2S^{-1} = 4Id$ and we get $D^2 = S^{-1}(4Id)S = 4IdS^{-1}S = 4Id$ as scalars and the identity commute with everything. So every entry on the diagonal of D is ± 2. However, recall that the trace of a matrix, that is, the sum of it's diagonal entries is unchanged by a change of basis. So we have a bunch of ± 2's whose sum is $1+1+1+1 = 4$. So 3 of these are 2 and 1 is a -2. Now, $\text{Det}A = \text{Det}S\text{Det}D\text{Det}S^{-1} = \text{DetD}$ (that is to say, determinants are also unaffected by change of basis. But $\text{Det}D$ is easy, it is $2*2*2*(-2)$.

(I've thought about how to approach this without using diagonaliaibility of diagonal matrices and traces, and I can't find a way I'm convinced is faster than expanding this to get the sign right.)

9)a) Completely trivial.

b) If $\text{Det}A \neq 0$, then by the above $A((1/\text{Det}A)\bar{A}) = Id$. If $\text{Det}A = 0$ but $AB = BA = Id$, then $\bar{A} = BA\bar{A} = B(\text{Det}A)Id = 0$. But clearly, if $\bar{A} = 0$, then $A = 0$, and 0 is not invertible.

c and d) Yes. Each term on the diagonal of $A\bar{A}$ simply corresponds to computing $\text{Det}A$ by a cofactor expansion on the corresponding row. For $i \neq j$, the ij^{th} entry of $A\bar{A}$ is the cofactor expansion you'd get if you computed the determinant of the matrix gotten from A by a replacing the i^{th} row by the j^{th} one (and leaving the j^{th} row the same. But this matrix has two identical rows, so its determinant is 0.

10) a) Writing out the product in question, we get

$$\begin{pmatrix} a_{11} & a_{11}x \\ a_{11}y & xy + DetA/a_{11} \end{pmatrix}$$

Clearly this will work if and only if $x = a_{12}/a_{11}$ and $y = a_{21}/a_{11}$.

b) As Det$A \neq 0$, a_{21} and $a_{12} \neq 0$ so SA has a nonzero entry in the upper left and by the preceeding can be expressed as $B\Lambda C$. But SA has a 0 in the lower left, so by the formulas in the previous part, $B = Id$ and $SA = \Lambda C$. As $S^2 Id$, $A = S\Lambda C$.

c) This problem is false, we need to require the determinant of the upper left corner to be nonzero as well.

Notice that the product of two upper or lower triangular matrices is always again upper or lower triangular, and the property of having 1's on the diagonal is preserved. Also, the inverse of such a matrix is such a matrix. This it suffice to show that by multiplying A on the left by lower and the right by upper triangular matrices we can get to a diagonal matrix Λ, as then we will have $BAC = \Lambda$ and $A = B^{-1}\Lambda C^{-1}$.

Also, notice that if A' is formed from A by taking a column of A', and adding x times it to some column to the right of that one, then $A' = AC$ for C upper triangular with a diagonal of 1's and similarly if we get A' by adding x times a row of A to a lower row, then we have $A' = BA$. Now, by adding suibtable multiples of the left hand column to the others, we can arrange that the top row if of the form $(*, 0, 0)$, where $*$ will reperesent a non zero number. We can then, by adding suitable multiples of this row to the other, arrange for the left column to be of this form.

As we reqruied the upperleft to have nonzero determinant, and we have effectively just multiplied it on each side by a matrix of determinant 1, it still has nonzero determinant, so the middle entry is nonzero and we can reeat this trick to eliminate the last two off-diagonal entries.

d) Find the top nonzero enry in the left column. By adding the row this is in to the rows below it, make all the other entries 0. Repeat in the next column, ignoring the row the previous number was in. Kepp going this way, form left to right, and each row will have it's first nonzero entry in a different column. Now add columns so there is only one nonzero entry in each row and column, this will be of the form $S\Lambda$. (If at any point a row of all zeroes is encountered, you can just leave it alone.