1a) Reversing the order of integration,

$$\int_0^x \int_0^s f(t)dtds = \int_0^x \int_t^x f(t)dsdt = \int_0^x (x-t)f(t)dt$$

.

b) Similarly, we get

$$\int_0^x \frac{(x-t)^2}{2} f(t)dt$$

.

2a) This is best done by symmetry: the function being integrated is odd with respect to y, the region of integration with respect to reflection over the x-axis, so the integral is 0.

b) $\int_0^2 \sin(r^2)2\pi r dr$. Setting $u = r^2$, $\int_0^4 \sin u\pi du = \pi(1 - \cos 4)$.

3a) Split this into the difference of an integral of f and of g. Do the f integral on x first, to get $\int_c^d f(b, y) - f(a, y)dy$. Similarly, do the integral on y first for g. We get

$$\int_c^d f(b,y)dy + \int_b^a g(x,d) + \int_d^c f(a,y) + \int_a^b g(x,c)$$

.

b) Integrating on x, we get $\int_c^d f_y(b, y) - f_y(a, y)$. Integrating again, we get the result.

4a)

$$\int_1^4 x^2 e^{tx^2} dx$$

.

b)

$$\int_t^{t^2} x^2 e^{xt^2} dx + 2te^{t^5} - e^{t^3}$$

.

c)

$$v_x = 2(g(2x+t) - g(2x-t)),\ v_t = (g(2x+t) + g(2x-t))$$
$$v_{xx} = 4(g'(2x+t) - g'(2x-t)),\ v_{tt} = g(2x+t) - g(sx-t).$$

5a) The integral is over the region $x > 0$, $x < 1-y^2$, $z^2 < x$. We can write this as $\int_0^1 dx \int_{z^2}^1 dz \int_0^{\sqrt{(1-x)}} dy$.

b) The integral is over $z < x^2+y^2$, $0 < x, y < 1$¿ This could be written as $\int_0^1 dz \int_0^1 dy \int_{\sqrt{(\min(0,z-x^2))}}^1 dx$.

6a and b) We show the volume of an n dimensional ball of radius a is given by $c_n a^n$ where $c_n = (\int_{-\pi/2}^{\pi/2} (\cos\theta)^n d\theta))k_{n-1}$. Call this number $V_n(a)$. Integrating on the n^{th} coordinate, we get

$$V_n(a) = \int_{-a}^{a} V_{n-1}(\sqrt{(a^2-x^2)})dx = \int_{-a}^{a} a_{n-1}(a^2-x^2)^{(n-1)/2}dx$$

by induction. Making the change of variable $x = a\sin\theta$

$$a_{n-1}\int_{-\pi/2}^{\pi/2} (a\cos\theta)^{n-1} a\cos\theta d\theta$$

.

So we have the claim. The only thing that remains is to evaluate $\int_{-\pi/2}^{\pi/2} (\sin\theta)^n d\theta$. There us a standard trick for such integrals: Call the value of this integral I_n.

$$I_n = \int_{-\pi/2}^{\pi/2} (\sin\theta)^n d\theta = \int_{-\pi/2}^{\pi/2} (1-\cos^2\theta)\sin^{n-2}\theta d\theta =$$

$$I_{n-2} - \int_{-\pi/2}^{\pi/2} (sin^{n-2}\theta \cos\theta)(\cos\theta d\theta)$$

.

Integrating by parts, we get

$$I_n = I_{n-2} - \frac{\sin^{n-1}\theta\cos\theta}{n-1}|_{\pi/2}^{\pi/2} + \frac{1}{n-1}\int_{-\pi/2}^{\pi/2} \sin^n\theta = I_{n-2} - I_n/(n-1)$$

.

So $I_n = I_{n-2}(n-1)/n$. We clearly have $I_0 = \pi$ and $I_1 = 2$ so

$$I_n = \frac{(n-1)(n-3)\cdots 1 * \pi}{n(n-2)\cdots 2},\ n \text{ even}$$

$$I_n = \frac{(n-1)(n-3)\cdots 2 * 2}{n(n-2)\cdots 3},\ n \text{ odd}$$

.

We now evaluate $c_n = I_n \dot{I}_{n-1} \cdots I_2 k_1$. A little thought shows $c_1 = 2$, so $c_n = I_n I_{n-1} \cdots I_1$. Note the helpful identity $I_n I_{n-1} = 2\pi/n$. If $n = 2k$, we get

$$c_n = (\frac{\pi}{n/2})(\frac{\pi}{(n-2)/2})\cdots(\frac{\pi}{2/2}) = \pi^k/k!$$

.

If $n = 2k+1$, we get

$$c_n = I_n c_{2k} = \frac{(2k)(2k-2)\cdots 2 * 2}{(2k+1)(2k-1)\cdots 3}\frac{\pi^k}{k!} =$$

$$\frac{2^k k! 2\pi^k}{(2k+1)(2k-1)\cdots 3k!} = \frac{(2\pi)^k 2}{(2k+1)(2k-1)\cdots 3}.$$

This can be rewritten more elegantly as

$$\frac{\pi^k}{(k+1/2)(k-1/2)\cdots(3/2)(1/2)}.$$

As a check, we give the first four values, $n = 4$ is part a. 2, π, $4\pi/3$, $\pi^2/2$.