1) Let x_0 be the point where we are checking this result. $d(h)(x_0)$ is just another name for the linear map $D_h|_{x_0} : V \to \mathbf{R}$. So we are being asked to show that

$$D_{fg}|_{x_0} = f(x_0)D_g|_{x_0} + g(x_0)D_f|_{x_0}$$

which is the standard product rule for multivariable functions. (If you're not familiar with this, notice that evalualting both sides of this expression on any vector v gives

$$\frac{\partial}{\partial v} fg|_{x_0} = f(x_0)\frac{\partial}{\partial v} g|_{x_0} + g(x_0)\frac{\partial}{\partial v} f|_{x_0}$$

which is the product rule for one variable.)

2) e^i is the linear map $e^i(x_i) = 1$, $e^i(x_j) = 0$, $i \neq j$. So we are being asked to show that $dx_i(x_j)$ has these properties. Unraveling the notation, we are being asked to determine

$$\frac{\partial x_i}{\partial x_j}$$

where the x_j is a variable we are differentiating with respect to and x_i is a function that assigns every point its i^{th} coordinate. The result is now clear.

3) Let me first make a comment on how this formula would look written out in terms of matrices for some particualr basis. Because ω is a map $V \to \mathbf{R}$ and not a member of V, this formula writes it as a row vector, or the matrix of a map $\mathbf{R}^n \to \mathbf{R}$. D_ϕ is then multiplied by on the right, and there are no transposes anywhere in the equation. If you remember this formula, you will need to remember that your vector is a row vectors and the matrix multiplying it acts on the right, both somewhat unusual things. An alternative approach is to take the transpose of everything. Now ω and $\phi^{*,1}(\omega)$ can be written as columns and you multiply on the left, but you need to remember to use D_ϕ^T.

If the spaces V and V' are of different dimensions, you will never have any trouble writing down the wrong formulae, as the wrong ones will give you impossible matrix multiplications. But when V and V' are the same dimension, or even worse the same space, it is easy to mix this up.

We know check parts (a) and (b) of the hint

a) This map is linear as ω and D_ϕ are linear. We have

$$\phi^{*,1}(f\omega)(x_0) = (f(\phi(x_0)) \cdot \omega(\phi(x_0))) \circ D_\phi|x_0.$$

(x_0 is the point in U' where we are checking the formula, we have not written in expilicitly the vectors u' and u.)

We can pull the $f(x_0)$ out of the parentheses to get

$$f(\phi(x_0)) \cdot (\omega(\phi(x_0)) \circ D_\phi|x_0)$$

which is what we wre asked to show.

To show $\phi^{*,1}(df) = d\phi^{*,0}f$, we note

$$\phi^{*,1}(df) = \phi^{*,1}(D_f) = D_f \circ D_\phi$$

and

$$d\phi^{*,0}f = d(f \circ \phi) = D_{f\circ\phi}.$$

So we are being asked to show $D_f \circ D_\phi = D_{f\circ\phi}$, which is just the multivariate chain rule.

b) We now check that $\phi^{*,1}$ is unique. For this purpose, we choose a coordinate system $x_1, x_2, \ldots x_n$ for U. Recall that any ω can be written as

$$\omega = \sum f_i dx_i$$

for some smooth functions f_i. If ψ is a linear map that obeys $\psi(f\omega) = (f\circ\phi)(\psi(\omega))$ and $\psi(df) = d(f\circ\phi)$, then using this relations in the order given we get

$$\psi(\omega) = \sum \psi(f_i dx_i) = \sum (f_i \circ \phi)(\psi dx_i) = \sum (f_i \circ \phi)(d(x_i \circ \psi)).$$

where $x_i \circ \psi$ is just the map that projects ψ onto thei^{th} coordinate. So we see $\psi(\omega)$ is uniquely determined and, as we have already found a map that obeys these properties, this must be that map.

4) a) We get $\sum t^i dt^i = \sum t^i (it^{i-1}) = \sum it^{2i-1}$.

b) We get $u^2 d(uv) + uvd(v^2) + v^2 d(u^2) = u^2(udv + vdu) + uv(2vdv) + v^2(2udu)$. This can be simplified a little as $(u^2v + 2uv^2)du + (u^3 + 2uv^2)dv$.

The second 4) a) The two parameteriations will have the same orientations if and only if they have the same endpoint as their beginning point and the same one as the ending point. This requires that $g(a) = a'$ and $g(b) = b'$. By the one to one nature of α and α', we have that g is one-to-one, so it is either increasing or decreasing. THe endpoints will work out iff it is increasing, in which case it's derivative is > 0 as required.

b) We expand each side of the expression in terms of f_{12} and f_{23}. To save writing in the following equations, we will not write down at what points on the c_i forms are being evaluated; take a point $u_1 \in C_1$ and it's corresponding images under f_{12} and f_{13}, forms should always be evaluated at whichever of these points makes sense.

$$f_{13}^{*,1}(\omega) = \omega \circ D_{f_{13}} = \omega \circ D_{f_{23}\circ f_{12}}.$$

$$f_{12}^{*,1}(f_{23}^{*,1}(\omega)) = f_{12}^{*,1}(\omega \circ D_{f_{23}}) = \omega \circ D_{f_{23}} \circ D_{f_{12}}.$$

So this problem comes down to the chain rule; $D_{g\circ h} = D_g \circ D_h$.

c) Let α and α' be the two parameterizations and let g be the function such that $\alpha' = \alpha \circ g$. (Part (a) gave permission to assume this function exists.) So, by part (b), $\alpha'^{*,1}(\omega) = g^{*,1}(\alpha^{*,1}(\omega))$. Write $\eta = \alpha^{*,1}\omega$. We are being asked to show

$$\int_a^b \eta = \int_{a'}^{b'} g^{*,1}\eta$$

which was exactly the lemma proven in class.

5) a) In each case, we parameterize the curve, record the pull back of the differential form, and evaluate it.

C_1 We parameterize the curve by $\alpha : t \to (t,t)$. So $\alpha^{*,1}dx = \alpha^{*,1}dy = dt$. So $\alpha^{*,1}(6x - y^2)dx - 2xydy = (6t - t^2)dt - 2t^2dt$. So $\int_{C_1} \omega = \int_0^1 6t - 3t^2 = 6/2 - 3/3 = 2$.

C_2. We use $\alpha : t \to (t, t^2)$. So $\alpha^{*,1}dx = dt$, $\alpha^{*,1}dy = 2tdt$. We get $\alpha^{*,1}(6x - y^2) - 2xy = (6t - t^4)dt - 2t \cdot t^2(2tdt)$. We get $\int_0^1 6t - t^4 - 4t^4 dt = 6/2 - 5/5 = 2$.

C_3. We break this curve into it's two parts and parameterize them by $\alpha : t \to (t, 0)$ and $\beta : t \to (1, t)$. So $\alpha^{*,1}dx = dt$, $\alpha^{*,1}dy = 0$, $\beta^{*,1}dx = 0$ and $\beta^{*,1}dy = 1$. So $\alpha^{*,1}(6x - y^2)dx - 2xydy = 6tdt$ and $\alpha^{*,1}(6x - y^2)dx - 2xydy = -2tdt$. So in total we have $\int_0^1 6t - \int_0^1 2t = 6/2 - 2/2 = 2$.

b) We have shown that the integrals of this form around three differnet curves are equal, and it is in fact true that the integral along any curve between these points is the same. Another way to say this is that the integral around a curve that goes out from $(0m0)$ to $(1,1)$ and back is 0. In fact, the integral around every curve of this form will be zero. By Green's theorem, the integral around any curve of ω is the integral over the area within of $d(6x - y^2)/dy - d(-2xy)/dx = (-2y) - (-2y) = 0$. So the integral around *any* curve of this form will be 0; and we get that for any pair of points, the integrals along two different paths joining those points will be the same.

6) The 1-form is

$$(1 + \cos \pi t)(-\pi \sin \pi t \; dt) + (\cos \pi t)(\sin \pi t)(\pi \cos \pi t \; dt) + (\sin \pi t + t)(dt).$$

Simplifying, we get

$$(\sin \pi t) - \pi(\sin \pi t)(\cos \pi t) - \pi(\sin \pi t)(1 - \cos^2 \pi t) + tdt$$

The integral is completley standard, except for recalling the indentity $(\sin\theta)(1 - \cos\theta) = \sin^3\theta = (1/4)\sin 3\theta - (3/4)\sin\theta$. We get $25/6 - 2/\pi$.

7) a) True. Using the parameterization $\alpha : t \to (c, t)$, we see that $\alpha^{*,1}dx = 0$ so $\alpha^{*,1}pdx = 0$.

b) False. Note that if this integral were always nonnegative then, by reversing the direction of the curve we'd have that it were also always nonpositive, hence always 0, which isn't true. An explicit counterexample is $p = q = 1$, C runs on a line segment from $(1,1)$ to $(0,0)$. It is true that if the line segemnt always has positive slope and is oriented in the direction of incressing both coordinates, this is true.

8) a) We use the parameterization $x = \cos r \sin 2\pi t$, $y = \sin r \sin 2\pi t$, $z = \cos 2\pi t$. It is easy to check this parameterizes the correct curve and nowhere has all derivatives vanish.

b) Note that any integral where $\sin 2\pi t$ or $\cos 2\pi t$ appears to an odd power will be 0. This eliminates the ydx and xdy terms immediately. We are left with

$$\int_X x - ydz = 2\pi \int_0^1 (-\cos r + \sin r) \cdot (\sin^2 2\pi t)dt$$

which is $(\sin r - \cos r)/2$.

We also must evaluate

$$\int_C z(dy - dx) = 2\pi \int_0^1 (\sin r - \cos r)/2 \cdot (\cos^2 2\pi t)dt = (\sin r - \cos r)/2$$

.

Combining these, we get our final answer of $\sin r - \cos r$.