1) This is the number of ways of choosing r distinct objects from a set of d. There are d choices for the first object, $d-1$ for the second, down to $d-r+1$ for the r^{th}. But we have counted each list $r!$ times, when in fact we only want them in increasing order, so the actual answer is

$$\frac{d(d-1)\cdots(d-r+1)}{r!} = \frac{d!}{(d-r)!r!}$$

2) a) It is obvious that ω^I is bilinear, antisymmetric and has the required values of $\omega^I(e_{i_1}, e_{i_2})$. Now, let η be any fomr with these properties. Then for any vectors $\sum a_i e_i$ and $\sum b_i e_i$, we have

$$\eta(\sum_i a_i e_i, \sum_i b_i e_i) = \sum_{i,j} a_i b_j \eta(e_i, e_j)$$

by bilinearity. As η is antisymmetric this is

$$\sum_{i<j} (a_i b_j - a_j b_i)\eta(e_i, e_j).$$

By the assumption on the values of $\eta(e_i, e_j)$, this must be $a_i b_j - a_j b_i$, which is precisely $\omega^I(\sum a_i e_i, \sum b_i e_i)$. $\blacksquare$ Note incidently that this argument shows in general that a member of $\Lambda^2(V^v)$ is determined by its values on the pairs (e_i, e_j), $i < j$.

b) Let η be any antisymmetric bilinear form, we must show it can be uniquely expressed as $\sum_{i<j} c_{ij}\omega^{(i,j)}$. The argument above shows that

$$\eta(\sum a_i e_i, \sum_i b_i e_i) = \sum_{i<j} (a_i b_j - a_j b_i)\eta(e_i, e_j).$$

This can be rewritten as

$$\eta(v, w) = \sum_{i<j} \omega^{(i,j)}(v, w)\eta(e_i, e_j).$$

This expression gives η as a sum of ω^I's. We now must show this expresison is unique. Let $\eta = \sum_{i<j} c_{ij}\omega^{(i,j)}$ be eny such expression. Applying both sides to e_i, e_j, we get $\eta(e_i, e_j) = c_{ij}$, so the represetnation is unique.

c) It is easy to check that

$$\omega^I(v_1, v_2, v_3) = e^{i_1}(v_1)e^{i_2}(v_2)e^{i_3}(v_3) + e^{i_1}(v_2)e^{i_2}(v_3)e^{i_3}(v_1) + e^{i_1}(v_3)e^{i_2}(v_1)e^{i_3}(v_2) - e^{i_1}(v_1)e^{i_2}(v_3)e^{i_3}(v_2) - e^{i_1}(v_3)e^{i_2}(v_2)e^{i_3}(v_1) -$$

has the required properties, and we must show it is unique. For any antisymetric bilinear form η, we have

$$\eta(\sum_i a_i e_i, \sum_j b_j e_j, \sum_k c_k e_k) = \sum_{i,j,k} a_i b_j c_k \eta(e_i, e_j, e_k).$$

Using the antisymmetry of η, we note that $\eta(i,j,k) = \eta(j,k,i) = \eta(k,i,j) = -\eta(i,k,j) = -\eta(k,j,i) = -\eta(j,i,k)$. If any of i, j, k are equal, then $\eta(i,j,k) = 0$ and if not we can reorder them uniquely such that $i < j < k$. Therefore, our expression equals

$$\sum_{i<j<k} (a_i b_j c_k + a_j b_k c_i + e_k b_i c_j - a_i b_k c_j - a_k b_j c_i - a_j b_i c_k)\eta(e_i, e_j, e_k).$$

So an antisymmetric form η is determined by the values $\eta(e_i, e_j, e_k)$, $i < j < k$. As we have specified the behavior of $\omega^{(i_1,i_2,i_3)}$ on these triples, $\omega^{(i_1,i_2,i_3)}$ is uniquely determined.

d) Our derivation above shows that for any antisymmetric, bilinear form η

$$\eta(u,v,w)=\sum_{i<j<k}\omega^{(i,j,k)}(u,v,w)\eta(e_i,e_j,e_k)$$

so the ω^I span Λ^3. If we have any representation $\eta=\sum c_{ijk}\omega^{(i,j,k)}$. we apply both sides to e_i,e_j,e_k and find $c_{ijk}=\eta(e_i,e_j,e_k)$, so the representation is unique. By part 1, the dimension of Λ^3 is $d(d-1)(d-2)/6$.

e) For any $i_1<i_2<\ldots<i_r$, set

$$\omega^{(i_1,\ldots,i_r)}(v_1,\ldots,v_r)=\sum_{\sigma\in S_r}\epsilon(\sigma)e^1(v_{\sigma(1)})\cdots e^r(v_{\sigma(r)}).$$

We have that the ω^I are a basis for Λ^r.

3 a) The linearity is obvious, we must show $\omega'\wedge\omega''$ is antisymmetric. Let τ be any permutation of the indices $1, ldots, r$. We have

$$\omega'\wedge\omega''(v_{\tau(1)},\ldots,v_{\tau(r)})=\sum_{\sigma\in S_r}\epsilon(\sigma)\omega(v_{\sigma\circ\tau(1)},\ldots,v_{\sigma\circ\tau(r')})\omega''(v_{\sigma\circ\tau(r'+1)},\ldots,v_{\sigma\circ\tau(r'+r'')})$$

which can be rewritten as

$$\sum_{\sigma\in S_r}\epsilon(\sigma\circ\tau)\epsilon(\tau^{-1})\omega(v_{\sigma\circ\tau(1)},\ldots,v_{\sigma\circ\tau(r')})\omega''(v_{\sigma\circ\tau(r'+1)},\ldots,v_{\sigma\circ\tau(r'+r'')})$$

using the key property $\epsilon(\sigma\circ\tau)=\epsilon(\sigma)\epsilon(\tau)$. AS $\epsilon(\tau)=\pm1$, $\epsilon(\tau^{-1})=\epsilon(\tau)$. If σ runs over all permutations in S_r, then so does $\sigma\circ\tau$, so our expression is

$$\epsilon(\tau)\omega'(v_1,\ldots_{r'})\wedge\omega''(v_{r'+1},\ldots,v_{r'+r''})$$

as desired.

b) Let i belong to both I' and I''. Let π be the permutation that switches the two appearances of i. Pair off the permutations σ and $\sigma\circ\pi$ in the sum; in each pair the two paried terms contribute exactly negatives of each other, as $\epsilon(\sigma\circ\pi)=\epsilon(\sigma)\epsilon(\pi)=-\epsilon(\sigma)$. So the terms of every pair cancel and we get 0.

c) The hard part of this problem is figuring out 1) where all the factorials go and 2) what to do with the $\pm$ factor.

We have

$$\omega^{I'}\wedge\omega^{I''}(v_1,\ldots,v_{r'+r''})=\frac{1}{r'!r''!}\sum_{\sigma}\epsilon(\sigma)\omega^{I'}(v_{\sigma(1)},\ldots,v_{\sigma(r')})\omega^{I''}(v_{\sigma(r'+1)},\ldots,v_{\sigma(r'+r'')})$$

Now, substituting the defintion of $\omega^{I'}$, we have

$$\frac{1}{r'!r''!}\sum_{\sigma}\epsilon(\sigma)(\sum_{\tau'}\epsilon(\tau')e^{i'_1}(v_{\sigma\circ\tau'1})\cdots e^{i'_{r'}}(v_{\sigma\circ\tau'r'}))(\sum_{\tau''}\epsilon(\tau'')e^{i''_1}(v_{\sigma\circ\tau''r'+1})\cdots e^{i''''_{r}}(v_{\sigma\circ\tau''r'+r''}))$$

where σ runs over permutations of 1, 2, ..., $r'+r''$, τ' over permutations of 1, ..., r' and τ'' over permutations of $r'+1,\ldots r'+r'$. Define j_k by $j_k=i'_k$, $1\le k\le r'$ and $j_k=i''_{k+r'}$, $r'+1\le k\le r''$. Viewing τ' as a permutation of all of 1, ..., $r'+r''$ that fixes the last r'' elements and τ'' as one that fixes the first r', we can write this as

$$\frac{1}{r'!r''!}\sum_{\sigma,\tau',\tau''}\epsilon(\sigma)\epsilon(\tau)\epsilon(\tau'')e^{j_1}(v_{\sigma\tau'\tau''1})\cdots e^{j_{r'+r''}}(v_{\sigma\tau'\tau''r'+r''})$$

This formula appears to have a peculiar assymetry: in the original formula τ' related to the first r' terms in the way that τ'' related to the last r'', but here the order of composition is $\sigma\tau'\tau''$ in both cases. This is an illusory assymetry: τ' and τ'' commute (proof left to reader) so we can write any one of the producs as either $\sigma\tau'\tau''$ or $\sigma\tau''\tau''$.

Using the identity $\epsilon(\sigma)\epsilon(\tau')\epsilon(\tau'') = \epsilon(\sigma\tau'\tau'')$, we have that this is

$$\frac{1}{r'!r''!}\sum_{\pi} e^{j_1}(v_{\pi 1})\cdots e^{j_{r'+r''}}(v_{\pi r'+r''})N_\pi$$

where N_π is the number of ways to write π in the form $\sigma\tau'\tau''$ where τ' fixes the last r'' terms and τ'' fixes the first r''. For any τ', τ'', there is a unique σ such that $\pi = \sigma\tau'\tau''$, so we get $N_\pi = r'!r''!$ for every π. So we have

$$\sum_{\pi} \epsilon(\pi)e^{j_1}(v_{\pi 1})\cdots e^{j_{r'+r''}}(v_{\pi r'+r''})$$

This is almost $\omega^{I'\cup I''}$. The only diifculty is that we might not have j_k in increasing order. The number of transpositions necesary to put them in increasing order will be $c(I', I'')$, so we finally get the claimed result.

4) Recall taht when a surface $Sigma$ is defined as the zero setof a function f, the tangent plane at $\sigma \in \Sigma$ is the Kernel of D_f at σ. So, at $(x_0, y_0, z_0) \in \Sigma$, the tangent plane is $\{(x, y, z), xx_0 + yy_0 + zz_0 = 0\}$.

a) Let $\sigma = (x_0, y_0, z_0) \in \Sigma$. Since $x_0^2 + y_0^2 + z_0^2 = 1$, at least one of them is not zero, say $z_0 \neq 0$. To show $\omega_\Sigma(\sigma)$ is not 0, we just must give a pair of vectors on which it is not zero that lie in the tengent space. It is clear that $v_1 = (z_0, 0, -x_0)$ and $v_2 = (0, z_0, y_0)$ lie in that tangent space, we have $\omega(v_1, v_2) = x_0((0)(z_0) - (-x_0)(z_0)) + y_0((-x_0)(0) - (z_0)(y_0)) + z_0((z_0)(z_0) - (0)(0))$. This can be simplified to $-z_0(x_0^2 + y_0^2 + z_0^2) = -z_0$. As we assumed $z_0 \neq 0$, we have shown $\omega_\Sigma \neq 0$ at σ. If one of the other coordinates is not zero but z_0 is, the proof is similar.

b) Considering the point $(0, 0, 1)$, we have shown that ω_Σ assigns a negative orientation to the ordered basis $(1, 0, 0)$, $(0, 1, 0)$. This is oriented clockwise as we look from the inside, so ω_Σ agrees witht he ordinary orientation. (Recall the convention that if you point the fingers of your right hand along the positive x axis and curl them towards the positive y axis, your thumb should point along the positive z axis.)

Book Problems)

15.1.2) a) $x_2(x_2dx_1 + \sin x_1dx_3 + 2dx_4) + 3(x_3^2dx_2 + x_2x_4dx_3) = x_2^2dx_1 + 3x_3^2dx_2 + x_2(x_4 + \sin x_1)dx_3 + 2x_2dx_4$

b) $2x_4(x_3^2dx_2 + x_2x_4dx_3) - x - 3^2(x_2x_4dx_3) = 2x_4x_3^2dx_2 + (2x_2x_4^2 - x_2x_3^2x_4)dx_3$

c) $2(x_2dx_1 + \sin x_1dx_3 + 2dx_4)((x_1 - x_4)dx_1 + x_2dx_4) = 2x_2^2dx_1dx_4 + (\sin x_1)(x_1 - x_4)(-1)dx_1dx_3 + x_2\sin x_1dx_3dx_4 - 2(x_1 - x_4)dx_1dx_4 = (-2x_1 + 2x_2^2 + 2x_4)dx_1dx_4 + (x_4 - x_1)\sin x_1dx_1dx_3 + x_2\sin x_1dx_3dx_4$

d) $(2(x_2dx_1 + \sin x_1dx_3 + 2dx_4) + x_4(x_3^2dx_2 + x_2x_4dx_3))(x_2x_4dx_3) = 2x_2x_3^2x_4dx_1dx_3 + x_2x_3^2x_4^2dx_2dx_3 - 4x_2x_4dx_3dx_4$

e) We have already found $\omega_1\omega_3$ in part c, so we use $\omega_1\omega_2\omega_3 = -(\omega_1\omega_3)\omega_2$ to get $((-2x_1 + 2x_2^2 + 2x_4)dx_1dx_4 + (x_4 - x_1)\sin x_1dx_1dx_3 + x_2\sin x_1dx_3dx_4)(x_3^2dx_2 + x_2x_4dx_3) = -(2x_1 + 2x_2^2 + 2x_4)(x_3^2)dx_1dx_2dx_4 - (2x_1 + 2x_2^2 + 2x_4)(x_2x_4)dx_1dx_3dx_4 - (x_4 - x_1)(\sin x_1)x_3^2dx_1dx_2dx_3 + x_2(\sin x_1)x_3^2dx_2dx_3dx_4$.

I'm sorry, I haven't got these last four done, but I don't want to hold up the solution set. I'll email them out if peole want.

15.2.3) a) $(a_{11}dx_1 + a_{21}dx_2)(a_{12}dx_1 + a_{22}dx_2) = a_{11}a_{22}dx_1dx_2 + a_{21}a_{12}dx_2dx_1 = (a_{11}a_{22} - a_{21}a_{12})dx_1dx_2$ where we haev used that $dx_id_i = 0$ and $dx_idx_j = -dx_jdx_i$.

b) Just the same as b.

15.2.4) Let $\omega \in$

14.1.1) The easiest way to parametrize a plane is to find a point (x_1, x_2, x_3) on it and two vectors (u_1, u_2, u_3) and (v_1, v_2, v_3); then $(p, q) \to (x_1 + pu_1 + qv_1, x_2 + pu_2 + qv_2, x_3 + pu_3 + qv_3)$ is a parameterization, where (p, q) ranges over $\mathbf{R}^2$.

a) $(p, q) \to (1 + 2p + 2q, p, -q)$.

b) $(p, q) \to (1 + p + q, 3 - 2p, 4 + 2q)$

c) $(p, q) \to (1 + p, 3 - 8p - 3q, 2 + 2p0q)$

d) $(p, q) \to (1 + 2p + 3q, -1 + 3p + q, 1 + 2p + 6q)$

For a parallelogram, if we use the same method as above but have (p, q) range over $0 \leq p, q \leq 1$, we get the parallelogram with a vertex at (x_1, x_2, x_3) and sides (u_1, u_2, u_3) and (v_1, v_2, v_3). For the particular problesm we are given,

e) $(p, q) \to (1 + q, -2 + 6p + 5q, 1 + 2p - 2q)$
f) $(p, q) \to (3 + p + 2q, 2 + 5p - 4q, 1 + 4p - 4q)$
For the next two problems, we use spherical coordinates. Our range is $0 \leq \theta \leq 2\pi$, $-\pi/2 \leq \phi \leq \pi/2$.
g) $(\theta, \phi) \to (2\cos\theta\cos\phi, 2\sin\theta\cos\phi, 2\sin\phi)$
h) $(\theta, \phi) \to (2\cos\theta\cos\phi, 3\sin\theta\cos\phi, 4\sin\phi)$
i) Let $0 \leq x, y$, $x + y \leq 1$. We have $(x, y) \to (x, y, 1 - x - y)$.
j) These are the points with $z^2 \leq 3/4$. Let $0 \leq \theta \leq 2\pi$, $-\sqrt{3}/2 \leq z \leq \sqrt{3}/2$. WE have $(\theta, z) \to ((1/2)\cos\theta, (1/2)\sin\theta, z)$.

14.1.3) For $(x, y) \in U$, we have that the line in question is all points of the form $(-2 + 2t, xt, yt)$. Requiring this to lie on $x^2 + y^2 + z^2 = 1$, we have $4 - 8t + 4t^2 + x^2t^2 + y^2t^2 = 1$ or

$$t = \frac{-8 \pm \sqrt{(64 - 4 \cdot 3(x^2 + y^2 + 4))}}{4 + x^2 + y^2} = \frac{-8 \pm \sqrt{(16 - 12x^2 - 12y^2)}}{4 + x^2 + y^2}.$$

As we want a positive result, we need to take the positive square root. So we get

$$(x, y) \to (-2 + 2t, xt, yt)$$

where

$$t = \frac{-8t + \sqrt{(16 - 12x^2 - 12y^2)}}{4 + x^2 + y^2}.$$

14.1.6) We want the image of D_α. This is the plane spanned by $(1, 0, \partial f/\partial x)$ and $(0, 1, \partial f/\partial y)$.