MATH 23B SECOND MIDTERM

due April 7

Let $A \subset \mathbb{R}^n$ be a compact subset. We denote by A^0 *the interior* of A. In other words $A^0 = \{a \in A | \exists \epsilon > 0$ such that $D_a(\epsilon) \subset A\}$ where $D_a(\epsilon) \subset \mathbb{R}^n$ is the ball of radius ϵ with the center at a. We define the *boundary* ∂A as the difference $\partial A = A - A^0$. We say that A *has smooth boundary* if ∂A is a smooth hypersurface.

For any subset A 0f $\mathbb{R}^n$ and any function f on A we denote by $f\chi_A$ the function on $\mathbb{R}^n$ such that $f\chi_A(x) = f(x)$ if $x \in A$ and $f\chi_A(x) = 0$ if $x \in \mathbb{R}^n - A$.

1. Show that for any compact set A with smooth boundary and any continuous function f on A the function $f\chi_A$ on $\mathbb{R}^n$ is integrable.

Let $D = \{r, \theta, \phi \in \mathbb{R}^3 | 0 \leq r < \infty, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$ and $S : D \to \mathbb{R}^3$ be a map given by

$$S(r, \theta, \phi) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi)$$

The map S is called *spherical coordinates*

2. a)Show that for any compact set $A \subset \mathbb{R}^3$ with smooth boundary such that $S^{-1}(A) \subset (0, \infty) \times (0, 2\pi) \times (0, \pi)$ we have

$$\int_A f(x, y, z) dx dy dz = \int_{S^{-1}(A)} \tilde{f}(r, \theta, \phi) dr d\theta d\phi$$

where $\tilde{f}(r, \theta, \phi) := f \circ S(r, \theta, \phi) r^2 cos\phi$.

b)Find $\int_A z^n dx dy dz$ where $A = \{(x, y, z) \in \mathbb{R}^3 | z \geq 0, x^2 + y^2 + z^2 \leq 1\}$.

c)Define an analog of sperical coordinates for $\mathbb{R}^4$ and formulate and solve the analog of the problem b).

d) Find the volume of the ball $B_n \in \mathbb{R}^n$ of radius 1.

e)Let $C_n = \{x = (x_1, ..., x_n) \in B_n | -1/100 < x_1 < 1/100\}$

Prove that $vol(C_n)/vol(B_n) > 99/100$ if $n > 10^{100}$

3. Let $I = [-a, a] \times ... \times [-a, a] \subset \mathbb{R}^n$ be a cube,$F : I \to \mathbb{R}^n$ be a continuously differentiable map such that $|DF(x) - Id| < \epsilon$ for all $X \in I$. Show that $F(I) \subset (1 + n^2\epsilon)I + F(0, ..., 0)$. In other words, show that for any $x \in I$ we have

$$F(x) \in [v_1 - (1+n^2\epsilon)a, v_1 + (1+n^2\epsilon)a] \times ... \times [v_n - (1+n^2\epsilon)a, v_n + (1+n^2\epsilon)a]$$

where $(v_1, ..., v_n) = F(0, ..., 0)$.

4.a)Let $f(t) : [0, \infty]$ be a monotonically decreasing function such that $f(t) \to 0$ for $t \to \infty$. Show that the integral $\int_0^\infty f(t)\sin(t)dt$ exists.

b)Show that the integral $\int_0^\infty \sin(t^2)dt$ exists.

5. Give a conjectural formulation of the Fubini theorem for improper integrals and prove or disprove your conjecture.

6. Let $f_1, f_2 : [0,1] \to \mathbb{R}$ be functions on $[0,1], f := f_1 + f_2$. Show that $\int_{[0,1]}^{+} f(x)dx \leq \int_{[0,1]}^{+} f_1(x)dx + \int_{[0,1]}^{+} f_2(x)dx$