MATH 23B SOLUTION SET 2

Problem 1: The definition as written is only determined up to a sign depending on the ordering chosen for the i_ks. So we make the additional assumption that the representation $I = \{i_1, ..., i_r\}$ has $i_1 < i_2 < ... < i_n$.

a) Since ω^I is a linear combination of r-linear forms, it is clearly an r-linear form. Thus we need only check that it is antisymmetric. We only have to check this on sets of basis elements $e_{j_1}, e_{j_2}, ..., e_{j_r}$ where $J = \{j_1, ..., j_r\}$ is a set in $\mathcal{A}(r, d)$ with an ordering. $\omega^I(e_J) = \omega^I(e_{j_1}, e_{j_2}, ..., e_{j_r})$ must be 0 if J is not a permutation of I since sum element of each product must then be zero. If J is such a permutation, then the sum will still only have 1 nonzero term, the one corresponding to σ s.t. $\sigma(j_k) = i_k$ for all k, and this term will have value $sign(\sigma)$. So if we consider J' which is J except with j_b and j_{b+1} flipped, we know that $\omega^I(e'_J) = sign(\sigma \circ s_b)$ where s_b is the simple reflection that switches elementes b and $b+1$. But since s_b has sign -1, we know that $\omega^I(e'_J) = -sign(\sigma) = -\omega^I(e_J)$.

b) We have the following three properties for the ω^Is:

1. There are $\binom{d}{r}$ of them.
2. They are linearly independent.

Proof: Assume there are λ_I s.t. $\sum_I \lambda_I \omega_I = 0$. Applying both sides to J as defined in part a), $\sum_I \lambda_I \omega_I(J) = 0$. But in a) we realized that $\omega_I(J) = 0$ unless J is a permutation of I, so there is only one term in the sum, call it I_J, the set corresponding to J, and it is 1 or -1 depending on the sign of the permutation. Hence $\lambda_{I_J} = 0$, and since we can do this for any J, all the λ_Is are 0.

3. They span the set of antisymmetric r-linear forms.

Proof: Let $f \in \Omega^r(V)$. Then consider the form $\sum_I f(e_I)\omega_I$, where $f(e_I) = f(e_{i_1}, ..., e_{i_r})$. Applied to any e_I, this is equal to $f(e_I)$ and thus by antisymmetry, this agrees with f on any set of basis elements $e_{j_1}, ..., e_{j_r}$. Hence $f = \sum_I f(e_I)\omega_I$ and we have found a linear combination for f.

Any two of these properties imply the result.

Problem 2: We need to show that

$$\Omega^r(A \circ B)(\omega)(v_1, ..., v_r) = (\Omega^r(B) \circ \Omega^r(A))(\omega)(v_1, ...v_r)$$

for all ω and v_i. The left side is just $\omega(A \circ B(v_1), ..., A \circ B(v_r))$. The right side can be simplified by noting that $\Omega^r(A)(\omega)$ is an r-linear antisymmetric form, so the $\Omega^r(B)$ can be brought in to get $\Omega^r(A)(\omega)(B(v_1), ..., B(v_r))$ on the right. Then the $\Omega^r(A)$ can also be brought in to get $\omega(A \circ B(v_1), ..., A \circ B(v_r))$.

Problem 3: The setup of this problem is relatively complex, even if the manipulations are not. We need to find an expression for $\Omega^r(A)(\omega^I)$ and decompose in it terms of the basis forms ω^I. This seems hard, but we know that if we apply some linear combination of ω_Is to $e_J = (e_{j_1}, ..., e_{j_r})$ for some fixed J a set in incrasing order, only one of the ω_Is matters, the one that is a permutation of J, and that gives a value of 1. In other words,

$$\Omega^r(A)(\omega^I)(e_J) = (\sum_I \lambda_I \omega_I)(e_J) = \lambda_J.$$

So we have

$$\begin{aligned} a_{J,I} &= \Omega^r(A)(\omega^I)(e_J) \\ &= \omega^I(Ae_{j_1}, ..., Ae_{j_r}) \\ &= \omega^I(\sum_k a_{k,j_1} e_k, ..., \sum_k a_{k,j_r} e_k) \\ &= \sum_{\sigma \in S_r} sign(\sigma) \prod_{1 \le h \le r} e^{i_{\sigma(h)}}(\sum_k a_{k,j_h} e_k) \\ &= \sum_{\sigma \in S_r} sign(\sigma) \prod_{1 \le h \le r} a_{i_{\sigma(h)}, j_h}. \end{aligned}$$

Problem 4:

a) We can express our polynomial as

$$\sum_{i,j \ge 0} a_{ij} x^i y^j = \sum_{i>0, j\ge 0} a_{ij} x^i y^j + \sum_{j \ge 0} a_{0j} y^j.$$

When we plug in $x = 0$, the first term becomes zero, so the second term is identically zero for all y. But in the first part, each term in the summation has at least one power of x, and so is a divisible by x.

b) Let $u = x - y$, $v = x + y$. If we write x as $(u + v)/2$ and y as $(v - u)/2$ we can plug these expressions into our polynomial and get a polynomial $f(u, v)$ s.t. $f(0, v) = 0$. By a) this means that f is divisible by u and hence the orginal polynomial is divisible by $x - y$.

Note: The theory of polynomials is very interesting and often falls under the heading of algebraic geometry. If you'd like, think about the following generalized question: if you have two polynomials in n variables s.t. $f(x_1, ..., x_n) = 0$ if $g(x_1, ..., x_n) = 0$ (but not necessarily the other way) what can you say about f and g? This is very hard, but there are easier special cases.